ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{2^{6n-1}(4n-3)}\binom{4n}{2n}=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$$

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$$S=\sum_{n=1}^{\infty}\frac{F_{2n}}{2^{6n-1}(4n-3)}\binom{4n}{2n}=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$$

$$S=2\sum_{n=1}^{\infty}\frac{F_{2n}}{(8)^{2n}(4n-3)}\binom{4n}{2n}=\sum_{n=1}^{\infty}\sum_{k=1}^{2}e^{\pi ink}\frac{F_n}{(8)^n(2n-3)}\binom{2n}{n}$$

Now, using the ordinary generating function of central binomial coefficients:

$$\sum_{n=0}^{\infty}x^n\binom{2n}{n}=\frac{1}{\sqrt{1-4x}}\Rightarrow\sum_{n=1}^{\infty}x^{2n-4}\binom{2n}{n}=\frac{1}{x^4\sqrt{1-4x^2}}-\frac{1}{x^4}\Rightarrow$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{x^{2n-3}}{2n-3}\binom{2n}{n}=-\frac{8x^2\sqrt{1-4x^2}+\sqrt{1-4x^2}-1}{3x^3}+C$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{x^{2n}}{2n-3}\binom{2n}{n}=-\frac{8x^2\sqrt{1-4x^2}+\sqrt{1-4x^2}-1}{3}\Rightarrow$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{x^{n}}{2n-3}\binom{2n}{n}=-\frac{8x\sqrt{1-4x}+\sqrt{1-4x}-1}{3}$$

And the Binet's formula for Fibonacci sequence: $F_n=\frac{1}{\sqrt{5}}(\varphi^n-(1-\varphi)^n)$, where $\varphi=\frac{1+\sqrt{5}}{2}$ is the golden ratio

$$\Rightarrow S=\frac{1}{\sqrt{5}}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\left(\left(\frac{\varphi e^{\pi ik}}{8}\right)^n-\left(\frac{(1-\varphi)e^{\pi ik}}{8}\right)^n\right)\frac{1}{2n-3}\binom{2n}{n}$$

$$=\frac{1}{\sqrt{5}}\sum_{k=1}^{2}\left(\begin{array}{c}\frac{1}{3}\left(e^{i\pi k}(1-\varphi)\sqrt{1+\frac{1}{2}e^{i\pi k}(\varphi-1)}+\sqrt{1+\frac{1}{2}e^{i\pi k}(\varphi-1)}-1\right)+\\+\frac{1}{3}\left(-e^{i\pi k}\varphi\sqrt{1-\frac{1}{2}e^{i\pi k}\varphi}-\sqrt{1-\frac{1}{2}e^{i\pi k}\varphi}+1\right)\end{array}\right)$$

$$=\frac{1}{3}\frac{1}{\sqrt{5}}\left(\sqrt{\frac{1}{2}\left(2\sqrt{5}+5\right)}-\frac{1}{\sqrt{2}}\right)=\frac{1}{3}\frac{1}{\sqrt{5}}\sqrt{\left(\sqrt{\frac{1}{2}\left(2\sqrt{5}+5\right)}-\frac{1}{\sqrt{2}}\right)^2}=$$

$$=\frac{1}{3}\frac{1}{\sqrt{5}}\sqrt{3+\sqrt{5}-\sqrt{2\sqrt{5}+5}}$$

$=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$, **hence proved.**