RMM - Calculus Marathon 2301 - 2400

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Proposed by

Daniel Sitaru – Romania, Ankush Kumar Parcha-India

Shirvan Tahirov-Azerbaijan, Vasile Mircea Popa-Romania

Pavlos Trifon-Greece, Nguyen Van Canh-Vietnam

Cosghun Memmedov-Azerbaijan, Khaled Abd Imouti-Syria

Abbaszade Yusif-Azerbaijan, Abdul Mukhtar-Nigeria

Bui Hong Suc-Vietnam, Srinivasa Raghava-India

Togrul Ehmedov-Azerbaijan, Hikmat Mammadov-Azerbaijan

Solutions by

Daniel Sitaru – Romania, Togrul Ehmedov-Azerbaijan

Ravi Prakash-India, Bedri Hajrizi-Kosovo, Amin Hajiyev-Azerbaijan

Samed Ahmedov-Azerbaijan, Adrian Popa-Romania

Ankush Kumar Parcha-India, Ahmed Salem-Tunisia

Samir HajAli-Syria, Khaled Abd Imouti-Syria, Yen Tung Chung-Taiwan

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Gbenga Ajeigbe-Nigeria, Kartick Chandra Betal-India

Mirsadix Muzefferov-Azerbaijan, Cosghun Memmedov-Azerbaijan

Nelson Javier Villaherrera Lopez-El Salvador

Noureddine Sima-Algeria

2301. Prove the below closed form

$$\int_0^{\infty}\frac{\tan^{-1}(x^2)}{1+x^2}dx+\frac{1}{2}\int_0^{\infty}\frac{\tan^{-1}(4x^2)}{1+4x^2}dx+\frac{1}{3}\int_0^{\infty}\frac{\tan^{-1}(9x^2)}{1+9x^2}dx+\cdots=\frac{\pi^4}{48}$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\infty}\frac{\tan^{-1}(x^2)}{1+x^2}dx+\frac{1}{2}\int_0^{\infty}\frac{\tan^{-1}(4x^2)}{1+4x^2}dx+\frac{1}{3}\int_0^{\infty}\frac{\tan^{-1}(9x^2)}{1+9x^2}dx+\cdots$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{\tan^{-1}(kx)^2}{1+(kx)^2}dx\Bigg]_{kx=y}=\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\infty}\frac{\tan^{-1}(y^2)}{1+y^2}dy$$

$$\int_0^{\infty}\frac{\tan^{-1}(y^2)}{1+y^2}dy=\int_0^{1}\frac{\tan^{-1}(y^2)}{1+y^2}dy+\int_1^{\infty}\frac{\tan^{-1}(y^2)}{1+y^2}dy$$

$$=\int_0^{1}\frac{\tan^{-1}(y^2)}{1+y^2}dy+\int_0^{1}\frac{\tan^{-1}\left(\frac{1}{y^2}\right)}{1+y^2}dy=\frac{\pi}{2}\int_0^{1}\frac{dy}{1+y^2}=\frac{\pi^2}{8}$$

$$I=\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\infty}\frac{\tan^{-1}(y^2)}{1+y^2}dy=\frac{\pi^2}{8}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{8}\zeta(2)=\frac{\pi^4}{48}$$

2302. Find:

$$\Omega=\int_0^{12\pi}\frac{x\left(3\cos\frac{x}{2}-2\sin\frac{x}{3}\right)}{1+\left(\sin\frac{x}{2}+\cos\frac{x}{3}\right)^2}dx$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$,

we get

$$\Omega=\int_0^{12\pi}\frac{(12\pi-x)\left[3\cos\left(6\pi-\frac{x}{2}\right)+2\cos\left(4\pi-\frac{x}{3}\right)\right]dx}{1+\left[\sin\left(6\pi-\frac{x}{2}\right)+\sin\left(4\pi-\frac{x}{3}\right)\right]^2}$$

$$=\int_0^{12\pi}\frac{(12\pi-x)\left[3\cos\frac{x}{2}+2\cos\frac{x}{3}\right]}{1+\left[-\sin\frac{x}{2}-\sin\frac{x}{3}\right]^2}dx=12\pi\Omega_1=\Omega$$

$$2\Omega=12\pi\Omega_1,\text{ or }\Omega=6\pi\Omega_1$$

where

$$\Omega_1=\int_0^{12\pi}\frac{3\cos\left(\frac{x}{2}\right)+2\cos\left(\frac{x}{3}\right)}{1+\left[\sin\frac{x}{2}+\sin\frac{x}{3}\right]^2}dx=6\int_0^{12\pi}\frac{\frac{1}{2}\cos\frac{x}{2}+\frac{1}{3}\cos\frac{x}{3}}{1+\left[\sin\frac{x}{2}+\sin\frac{x}{3}\right]^2}dx$$

$$=6\left(\tan^{-1}\left[\sin\frac{x}{2}+\sin\frac{x}{3}\right]\right)\Big]_0^{12\pi}=6[0]=0$$

Thus, $\Omega=0$

2303. Solve for real numbers:

$$9\Omega^2(x)+9\Omega(x)+2=0,\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$\Omega(x)=\int_1^x\frac{t^{\frac{1}{2}}dt}{\sqrt{1+\left(t^{\frac{3}{2}}\right)^2}}=\left|\begin{matrix}t^{\frac{3}{2}}=u\\ \frac{2}{3}t^{\frac{1}{2}}dt=du\end{matrix}\right|=\frac{2}{3}\int_{u_1}^{u_2}\frac{du}{\sqrt{1+4^2}}=$$

$$=\frac{2}{3}\ln\left(t^{\frac{3}{2}}+\sqrt{1+t^3}\right)|_1^x=$$

$$=\frac{2}{3}\ln\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\frac{2}{3}\ln(1+\sqrt{2})=\frac{2}{3}\ln\frac{x^{\frac{3}{2}}+\sqrt{1+x^3}}{1+\sqrt{2}}$$

$$\Omega(x)=\frac{-9\pm\sqrt{9}}{18}=\frac{-9\pm3}{18}=\frac{-3\pm1}{6}=\left\langle\begin{matrix}-\frac{2}{3}\\ \frac{1}{3}\end{matrix}\right.$$

1) $\ln\frac{x\sqrt{x}+\sqrt{1+x^3}}{1+\sqrt{2}}=-1,\underbrace{x\sqrt{x}+\sqrt{1+x^3}}_{\ge1}=\underbrace{\frac{(1+\sqrt{2})}{e}}_{<1}$, no solution!

2) $\ln\frac{x\sqrt{x}+\sqrt{1+x^3}}{1+\sqrt{2}}=\frac{1}{2}$

$$x\sqrt{x}+\sqrt{1+x^3}=(1+\sqrt{2})\sqrt{e}$$

Let $(1+\sqrt{2})\sqrt{e}=a,\ \sqrt{1+x^3}=a-\sqrt{x^3}|^2$

$$1+x^3=a^2-2a\sqrt{x^3}+x^3$$

$$2a\sqrt{x^3}=a^2-1,\qquad x^3=\frac{a^2-1}{2a},\qquad x=\sqrt[3]{\frac{a^2-1}{2a}}$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$9\Omega^2(x)+9\Omega(x)+2=0;\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt$$

$$\left(\Omega(x)+\frac{2}{3}\right)\left(\Omega(x)+\frac{1}{3}\right)=0\to\Omega_1(x)=-\frac{2}{3}\quad\Omega_2(x)=-\frac{1}{3}$$

$$\Omega(x)=\int_1^x\frac{\sqrt{t}}{\sqrt{1+t^3}}dt\therefore\left(t^3=u\ \frac{du}{dt}=3t^2=3u^{\frac{2}{3}}\right)$$

$$\Omega(x)=\frac{1}{3}\int_1^{x^3}\frac{1}{\sqrt{u+u^2}}du\therefore\left(\sqrt{u}=v\ \frac{dv}{du}=\frac{1}{2v}\ \ v\in[x\sqrt{x};1]\right)$$

$$\Omega(x)=\frac{1}{3}\int_1^{x\sqrt{x}}\frac{2v}{\sqrt{v^2+v^4}}dv=\frac{2}{3}\int_1^{x\sqrt{x}}\frac{1}{\sqrt{1+v^2}}dv=$$

$$=\frac{2}{3}\Big|_1^{x\sqrt{x}}\sinh^{-1}(v)=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)$$

$$\begin{cases}-\frac{2}{3}=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)\\-\frac{1}{3}=\frac{2}{3}\left(\sinh^{-1}(x\sqrt{x})-\sinh^{-1}(1)\right)\end{cases}\begin{cases}\sinh^{-1}(x\sqrt{x})=\sinh^{-1}(1)-1\\\sinh^{-1}(x\sqrt{x})=\sinh^{-1}(1)-\frac{1}{2}\end{cases}$$

Answer: $x_1=\sinh^{\frac{2}{3}}(\sinh^{-1}(1)-1),\ x_2=\sinh^{\frac{2}{3}}\left(\sinh^{-1}(1)-\frac{1}{2}\right)$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(x)=\int_1^x\sqrt{\frac{t}{1+t^3}}dt=\int_1^x\frac{t^{\frac{1}{2}}}{\sqrt{1+\left(t^{\frac{3}{2}}\right)^2}}dt$$

$$\text{Put } t^{\frac{3}{2}}=u\Rightarrow\frac{3}{2}t^{\frac{1}{2}}dt=du$$

$$\therefore\Omega(x)=\frac{2}{3}\int_1^{x^{\frac{3}{2}}}\frac{du}{\sqrt{1+u^2}}=\frac{2}{3}\log\left(u+\sqrt{1+u^2}\right)\Big]_1^{x^{\frac{3}{2}}}$$

$$=\frac{2}{3}\left[\log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right)\right] \quad (1)$$

Now, $9\Omega(x)^2+9\Omega(x)+2=0 \Rightarrow (3\Omega(x)+1)(3\Omega(x)+2)=0$

$$\Rightarrow \Omega(x)=-\frac{1}{3} \text{ or } \Omega(x)=-\frac{2}{3} \quad (2)$$

$$\frac{2}{3}\left[\log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right)\right]=-\frac{1}{3},-\frac{2}{3}$$

[from (1) and (2)]

$$\Rightarrow \log\left(x^{\frac{3}{2}}+\sqrt{1+x^3}\right)-\log\left(1+\sqrt{2}\right)=-\frac{1}{2},-1$$

$$\Rightarrow \frac{x^{\frac{3}{2}}+\sqrt{1+x^3}}{1+\sqrt{2}}=\frac{1}{\sqrt{e}},\frac{1}{e} \Rightarrow \sqrt{1+x^3}+x^{\frac{3}{2}}=\frac{(\sqrt{2}+1)}{\sqrt{e}},\frac{\sqrt{2}+1}{e} \quad (3)$$

$$\Rightarrow \frac{(1+x^3)-x^3}{\sqrt{1+x^3}-x^{\frac{3}{2}}}=\frac{\sqrt{2}+1}{\sqrt{e}},\frac{\sqrt{2}+1}{e} \Rightarrow \sqrt{1+x^3}-x^{\frac{3}{2}}=\frac{\sqrt{e}}{\sqrt{2}+1},\frac{e}{\sqrt{2}+1} \quad (4)$$

Subtracting (4) from (3) we get

$$2x^{\frac{3}{2}}=\frac{\sqrt{2}+1}{\sqrt{e}}-\frac{\sqrt{e}}{\sqrt{2}+1},\frac{\sqrt{2}+1}{e}-\frac{e}{\sqrt{2}+1}$$

$$\Rightarrow 2x^{\frac{3}{2}}=\frac{\left(\sqrt{2}+1\right)^2-e}{\sqrt{e}\left(\sqrt{2}+1\right)},\frac{\left(\sqrt{2}+1\right)^2-e^2}{\left(\sqrt{2}+1\right)e}$$

$$\text{But } \frac{\left(\sqrt{2}+1\right)^2-e^2}{\left(\sqrt{2}+1\right)e}<0$$

$$x^{\frac{3}{2}}=\frac{\left(\sqrt{2}+1\right)^2-e}{\sqrt{e}\left(2\sqrt{2}+2\right)} \Rightarrow x=\left[\frac{\left(\sqrt{2}+1\right)^2-e}{\sqrt{e}\left(2\sqrt{2}+2\right)}\right]^{\frac{2}{3}}$$

2304. Prove that

$$I=\int_0^1\int_0^1\left(\frac{x-y}{x+y}\right)\frac{1}{\log\left(\frac{x}{y}\right)}dx\,dy=\log\left(\frac{\pi}{2}\right)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)\frac{1}{\log\left(\frac{x}{y}\right)}dx\,dy=\int_0^1 y\int_0^{\frac{1}{y}}\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz\,dy\overset{IBP}{=}\left[\frac{y^2}{2}\int_0^{\frac{1}{y}}\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz\right]_{y=0}^{y=1}-$$

$$\frac{1}{2}\int_0^1\left(\frac{1-y}{1+y}\right)\frac{1}{\log(y)}dy=\frac{1}{2}\int_0^1\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz-\frac{1}{2}\int_0^1\left(\frac{1-y}{1+y}\right)\frac{1}{\log(y)}dy=\int_0^1\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz=$$

$\int_0^1\int_0^1 \frac{z^m}{1+z} dm\, dz = \sum_{k=0}^{\infty}(-1)^k \int_0^1\int_0^1 z^{m+k}\, dz\, dm = \sum_{k=0}^{\infty}(-1)^k (\log(k+2) - \log(k+1)) = \sum_{k=1}^{\infty}(-1)^{k-1} \log\left(\frac{k+1}{k}\right) = 2\eta'(0) = \log\left(\frac{\pi}{2}\right)$. *Note*: $\eta'(0) = -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\log\left(\frac{k}{k+1}\right)$

2305. $f: \mathbb{R} \to \mathbb{R}, f(f(x)) + f(x) = -x, \forall x \in \mathbb{R}$

Find:

$$\Omega = \int_0^{\pi} f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right) dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Samed Ahmedov-Azerbaijan

$$f(f(x)) + f(x) = -x, \qquad f\left(f(f(x))\right) + f(f(x)) = -f(x)$$

$$f\left(f(f(x))\right) + f(f(x)) + f(x) = 0, \qquad f\left(f(f(x))\right) - x = 0$$

$$f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right) - \frac{1}{2+\cos x} \Longrightarrow f\left(f(f(x))\right) = \frac{1}{2+\cos x}$$

$$\Omega = \int_0^{\pi} \frac{1}{2+\cos x} dx, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\Omega = \int_0^{\pi} \frac{1}{2 + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx = \int_0^{\pi} \frac{1}{\frac{2+2\tan^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx = \int_0^{\pi} \frac{1+\tan^2\frac{x}{2}}{3+\tan^2\frac{x}{2}} dx$$

$$1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2} \Rightarrow \Omega = \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}} dx$$

$$\tan\frac{x}{2} = m \;\; x = 0, m = 0. x = \pi m = +\infty dx = \frac{2dm}{\sec^2\frac{x}{2}}$$

$$\Omega = \int_0^{+\infty} \frac{\sec^2\frac{x}{2}}{3+m^2} \cdot \frac{2}{\sec^2\frac{x}{2}} dm = \int_0^{+\infty} \frac{2}{(\sqrt{3})^2 + m^2} dm$$

$$\Omega = \lim_{\eta\to+\infty} \int_0^{\eta} \frac{2}{(\sqrt{3})^2 + m^2} dm = \lim_{\eta\to+\infty} \left(\frac{2}{\sqrt{3}} \arctan\frac{m}{\sqrt{3}}\right)\Bigg|_0^{\eta} = \lim_{\eta\to+\infty} \left(\frac{2}{\sqrt{3}} \arctan\frac{\eta}{\sqrt{3}} - 0\right) =$$

$$= \frac{\pi\sqrt{3}}{3} = \frac{\pi}{\sqrt{3}}, \qquad \Omega = \frac{\pi}{\sqrt{3}}$$

Solution 2 by Adrian Popa-Romania

$$f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)=-f\left(\frac{1}{2+\cos x}\right)-f\left(f\left(\frac{1}{2+\cos x}\right)\right)=$$

$$=-f\left(\frac{1}{2+\cos x}\right)-\left(-\frac{1}{2+\cos x}-f\left(\frac{1}{2+\cos x}\right)\right)=\frac{1}{2+\cos x}$$

$$\Omega=\int_0^{\pi}\frac{1}{2+\cos x}dx=\int_0^{\pi}\frac{1}{2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx=\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

$$\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx=dt$$

$$\left(1+\tan^2\frac{x}{2}\right)dx=2dt$$

$$x=0\Rightarrow t=0$$

$$x=\pi\Rightarrow t\to\infty$$

$$\Omega=\int_0^{\infty}\frac{2dt}{3+t^2}=2\cdot\frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}}\Big|_0^{\infty}=\frac{2}{\sqrt{3}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{3}}$$

Solution 3 by Ankush Kumar Parcha-India

We have, $f:\mathbb{R}\to\mathbb{R},(f\circ f)x+f(x)=x,\forall x\in\mathbb{R}$

$$\overset{x\to f(x)}{\Longrightarrow}(f\circ f\circ f)x+(f\circ f)x=-f(x)\underset{above}{\overset{From}{\Longrightarrow}}(f\circ f\circ f)(x)=x\qquad(1)$$

$$\underset{(1)}{\overset{Equation}{\Longrightarrow}}\int_0^{\pi}(f\circ f\circ f)\left(\frac{1}{2+\cos x}\right)dx=\int_0^{\pi}\frac{dx}{2+\cos(x)}$$

$$\overset{x\to\pi-x}{\Longrightarrow}\int_0^{\pi}\frac{dx}{2+\cos(\pi-x)}=\int_0^{\pi}\frac{dx}{2-\cos(x)}$$

$$\Rightarrow 2\int_0^{\pi}\frac{dx}{4-\cos^2(x)}=2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\cos^2(x)}+2\underbrace{\int_{\frac{\pi}{2}}^{\pi}\frac{dx}{4-\cos^2(x)}}_{x\to\frac{\pi}{2}+x}$$

$$\overset{\because\cos\left(x+\frac{\pi}{2}\right)=-\sin(x)}{\Longrightarrow}2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\cos^2(x)}+2\int_0^{\frac{\pi}{2}}\frac{dx}{4-\sin^2(x)}$$

$$\overset{\because\tan^2(x)+1=\sec^2(x)}{\Longrightarrow}2\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{4\tan^2(x)+3}dx+2\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{3\tan^2(x)+4}dx$$

$$\overset{\tan(x)\to x}{\Longrightarrow}2\int_0^{\infty}\frac{dx}{4x^2+3}+2\int_0^{\infty}\frac{dx}{3x^2+4}=\left(\frac{\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{x\sqrt{3}}{2}\right)}{\sqrt{3}}\right)_0^{\infty}$$

$$=\frac{\pi}{2\sqrt{3}}+\frac{\pi}{2\sqrt{3}}\Rightarrow\int_0^{\pi}(f\circ f\circ f)\left(\frac{1}{2+\cos x}\right)dx=\frac{\pi}{\sqrt{3}}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$f(f(x))+f(x)=-x \qquad (1)$$

$$\Rightarrow f\left(f(f(x))\right)+f(f(x))=-f(x)\Rightarrow f\left(f(f(x))\right)+f(f(x))+f(x)=0$$

$$\Rightarrow f\left(f(f(x))\right)-x=0 \quad \text{[using (1)]}\Rightarrow f\left(f(f(x))\right)=x$$

$$\therefore \Omega=\int_0^{\infty}\frac{dx}{2+\cos x}=\int_0^{\pi}\frac{dx}{1+2\cos^2\left(\frac{x}{2}\right)}=\int_0^{\pi}\frac{\sec^2\left(\frac{x}{2}\right)dx}{\tan^2\left(\frac{x}{2}\right)+3}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)\Big]_0^{\pi}=\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\pi}{2}\right)=\frac{\pi}{\sqrt{3}}$$

2306. Prove that

$$\int_0^1\int_0^{\infty}\frac{\log^2(x)\log^2(1+y^2)}{y(1+x^2)}dx\,dy=\frac{\pi^3}{64}\zeta(3)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^{\infty}\frac{\log^2(x)\log^2(1+y^2)}{y(1+x^2)}dx\,dy=\int_0^{\infty}\frac{\log^2(x)}{1+x^2}dx\int_0^1\frac{\log^2(1+y^2)}{y}dy$$

$$I_1=\int_0^{\infty}\frac{\log^2(x)}{1+x^2}dx=\int_0^1\frac{\log^2(x)}{1+x^2}dx+\int_1^{\infty}\frac{\log^2(x)}{1+x^2}dx=2\int_0^1\frac{\log^2(x)}{1+x^2}dx$$

$$=2\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{2k}\log^2(x)\,dx=4\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+2k)^3}=\frac{\pi^3}{8}$$

$$I_2=\int_0^1\frac{\log^2(1+y^2)}{y}dy\Bigg]_{y^2=z}=\frac{1}{2}\int_0^1\frac{\log^2(1+z)}{z}dz\overset{IBP}{=}-\int_0^1\frac{\log(x)\log(1+x)}{1+x}dx$$

$$=\frac{1}{8}\zeta(3)$$

$$I=I_1I_2=\frac{\pi^3}{64}\zeta(3)$$

NOTE:

$$\int_0^1\frac{\log(x)\log(1+x)}{1+x}dx=-\frac{1}{8}\zeta(3)\ \text{and}\ \sum_{k=0}^{\infty}\frac{(-1)^k}{(1+2k)^3}=\frac{\pi^3}{32}$$

2307. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^{2\pi} \sqrt{\sum_{k=1}^{n} \cos^2(kx)\, dx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ahmed Salem-Tunisia

$$\sum_{k=1}^{n} \cos^2 kx = \frac{1}{2}\left(\sum_{k=1}^{n} [2\cos^2(kx) - 1 + 1]\right) = \frac{n}{2} + \sum_{k=1}^{n} \cos(2kx) =$$

$$= \frac{n}{2}\left(1 + \frac{2}{n}\sum_{k=1}^{n} \cos(2kx)\right)$$

For $\alpha \geq 0, 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2 \leq \sqrt{1+\alpha} \leq 1 + \frac{1}{2}\alpha$

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}} \int_0^{2\pi} \left[1 + \frac{1}{n}\sum_{k=1}^{n} \cos(2kx) - \frac{1}{2n^2}\left(\sum_{k=1}^{n} \cos^2(2kx)\right)\right] dx \leq \Omega \leq$$

$$\leq \lim_{n\to\infty} \frac{1}{\sqrt{2}} \int_0^{2\pi} \left(1 + \frac{1}{n}\right) \sum_{k=1}^{n} \cos(2kx)\, dx$$

$$\int_0^{2\pi} \cos(2kx)\, dx = 0\ \forall k \in \mathbb{Z}_{\geq 1}$$

$$\int_0^{2\pi} \cos(2kx)\cos(2mx)\, dx = \begin{cases} \pi & if\ k = m \\ 0 & if\ k \neq m \end{cases}$$

$$\sqrt{2}\pi - \frac{1}{2\sqrt{2}} \lim_{n\to\infty} \frac{1}{n^2} n\pi \leq \Omega \leq \sqrt{2}\pi \Rightarrow \Omega = \sqrt{2}\pi$$

Solution 2 by Samir HajAli-Syria

$$\sum_{k=1}^{n} \cos^2(kx) = \sum_{k=1}^{n} \frac{1 + \cos\big(k(2x)\big)}{2} =$$

$$\sum_{k=1}^{n} \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{n} \cos\big(k(2x)\big) = \frac{1}{2}n + \frac{1}{2} \cdot \frac{\sin\left(\frac{n}{2}2x\right) \cdot \cos\left(\frac{n+1}{2}2x\right)}{\sin\left(\frac{2x}{2}\right)} =$$

$$\frac{1}{2}n + \frac{1}{2} \cdot \frac{\sin(nx) \cdot \cos(nx + k)}{\sin x} = \frac{1}{2}n + \frac{1}{2\sin x} \times \frac{1}{2}[\sin(2nx + x) + \sin(-x)] =$$

$$\frac{1}{2}n+\frac{1}{4\sin x}[\sin((2n+1)x)-\sin x]$$

$$\Rightarrow \frac{\sum_{k=1}^{n}\cos^2(kx)}{n}=\frac{1}{2}+\frac{1}{4\sin x}\left[\frac{\sin((2n+1)x)}{n}-\frac{\sin x}{n}\right]$$

Let put: $f(n,x):=\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}$

Then:

$$\Omega=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}$$

$$\lim_{n\to+\infty}\int_0^{2\pi}\sqrt{\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}}=\int_0^{2\pi}\sqrt{\lim_{n\to+\infty}f(n,x)}\,dx=\int_0^{2\pi}\sqrt{\frac{1}{2}}dx=\frac{2\pi}{\sqrt{2}}=\pi\sqrt{2}$$

where: $\lim_{n\to+\infty}\frac{\sin x}{n}=0$ and

$$\lim_{n\to+\infty}-\frac{1}{n}\le\lim\frac{\sin((2n+1)x)}{n}\le\lim_{n\to+\infty}\frac{1}{n}\Rightarrow\lim_{n\to+\infty}\frac{\sin((2n+1)x)}{n}=0$$

So, $\lim_{n\to+\infty}f(n,x)=\frac{1}{2}+\frac{1}{4\sin x}(0-0)=\frac{1}{2}$

Solution 3 by Khaled Abd Imouti-Syria

$$\sum_{k=0}^{n}\cos^2(kx)=\frac{1}{4}\left(3+2n+\frac{\sin x\,(1+2n)}{\sin x}\right)$$

$$\frac{\sum_{k=0}^{n}\cos^2(kx)}{n}=\frac{\frac{3}{4}+\frac{1}{2}n+\frac{1}{4}\frac{\sin(x(1+2n))}{\sin x}}{n}$$

$$\frac{1+\sum_{k=1}^{n}\cos^2(kx)}{n}=\frac{3}{4n}+\frac{1}{2}+\frac{1}{4n}\cdot\frac{\sin(x(1+2n))}{\sin x}$$

$$\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}=-\frac{1}{4n}+\frac{1}{2}+\frac{1}{4n}\cdot\frac{\sin(x(1+2n))}{\sin x}$$

$$\lim_{n\to+\infty}\left(\frac{\sum_{k=1}^{n}\cos^2(kx)}{n}\right)=\frac{1}{2},\qquad \Omega=\int_0^{2\pi}\frac{1}{\sqrt{2}}dx=\frac{1}{\sqrt{2}}\cdot 2\pi=\sqrt{2}\pi$$

Solution 4 by Amin Hajiyev-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}f_n^{\frac{1}{2}}(x)\,dx$$

$$f_n(x)=\sum_{k=1}^{n}\cos^2(kx)=\frac{1}{2}\sum_{k=1}^{n}(1+\cos(2kx))=\frac{n}{2}+\frac{1}{2}\mathcal{R}\sum_{k=1}^{n}e^{2ikx}=$$

$$=\frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\frac{e^{2ix}(e^{2inx}-1)}{e^{2ix}-1}\right\}=\frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\frac{e^{2ix}\times e^{ikx}(e^{inx}-e^{-inx})}{e^{ix}(e^{ix}-e^{-ix})}\right\}=$$

$$=\frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{e^{ix(n+1)}\frac{\sin(nx)}{\sin(x)}\right\}=\frac{n}{2}+\frac{1}{2}\mathcal{R}\left\{\left(\cos\left(x(n+1)\right)+i\sin\left(x(n+1)\right)\right)\frac{\sin(nx)}{\sin(x)}\right\}=$$

$$=\frac{n}{2}+\frac{1}{2}\cos\left(x(n+1)\right)\frac{\sin(nx)}{\sin(x)}=\frac{n}{2}+\frac{\cos(xn)\sin(xn)\cos(x)}{2\sin(x)}-\frac{\sin^2(xn)}{2}$$

$$\Omega=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{f_n(x)}{n}}\,dx=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}+\frac{\sin(2nx)}{4n}\tan(x)-\frac{\sin^2(nx)}{2n}}\,dx=$$

$$=\int_0^{2\pi}\sqrt{\lim_{n\to\infty}\left(\frac{1}{2}\right)+\frac{1}{4}\lim_{n\to\infty}\frac{\sin(2nx)}{n}\cot(x)-\frac{1}{2}\lim_{n\to\infty}\frac{\sin^2(nx)}{n}}\,dx;$$

$$0<x<2\pi\ \left\{n=\frac{1}{t};t\to 0\right\}$$

$$\Omega=\int_0^{2\pi}\sqrt{\frac{1}{2}+\frac{1}{4}\lim_{t\to 0}t\sin\left(\frac{2x}{t}\right)\cot(x)-\frac{1}{2}\lim_{t\to 0}t\sin^2\left(\frac{x}{t}\right)}\,dx=\int_0^{2\pi}\sqrt{\frac{1}{2}}\,dx=\pi\sqrt{2}$$

Answer:

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}\,dx=\pi\sqrt{2}$$

Solution 5 by Adrian Popa-Romania

$$\sum_{k=1}^{n}\cos^2 kx=\sum_{k=1}^{n}\frac{1+\cos 2kx}{2}=\sum_{k=1}^{n}\frac{1}{2}+\sum_{k=1}^{n}\frac{\cos 2kx}{2}=$$

$$=\frac{n}{2}+\frac{1}{2}\cdot\frac{\sin nx\cdot\cos(n+1)x}{\sin x}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{2\pi}\sqrt{\sum_{k=1}^{n}\cos^2(kx)}\,dx=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{n}{2n}+\frac{1}{2}\cdot\frac{\sin nx\cdot\cos(n+1)x}{n\sin x}}\,dx$$

$$=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}\left(1+\frac{\sin nx\cdot\cos(n+1)x}{n\sin x}\right)}$$

$$=\lim_{n\to\infty}\int_0^{2\pi}\sqrt{\frac{1}{2}\left(1+\frac{\sin(2n+1)x-\sin x}{2n\sin x}\right)}\,dx$$

$$=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\int_0^{2\pi}\sqrt{1+\underbrace{\frac{\sin(2n+1)x}{n\sin x}}_{\to 0}-\underbrace{\frac{1}{2n}}_{\to 0}}\,dx=\frac{1}{\sqrt{2}}\int_0^{2\pi}dx=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\pi$$

2308. If $I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz$ then, show that:

$$I=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$

where, $\zeta(3)$ is an Apery's constant.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=\int_0^1\int_0^1\int_0^1\log(xyz)\tan^{-1}(xyz)\,dx\,dy\,dz$$

$$=\int_0^1\frac{1}{x}\int_0^1\frac{1}{y}\int_0^{xy}\log(m)\tan^{-1}(m)\,dm\,dy\,dx\overset{IBP}{=}$$

$$-\int_0^1\log(y)\int_0^1\log(xy)\tan^{-1}(xy)\,dx\,dy$$

$$=-\int_0^1\frac{\log(y)}{y}\int_0^y\log(p)\tan^{-1}(p)\,dp\,dy\overset{IBP}{=}\frac{1}{2}\int_0^1\log^3(y)\tan^{-1}(y)\,dy\overset{IBP}{=}-\frac{3\pi}{4}$$

$$-\frac{1}{2}\left\{\int_0^1\frac{y\log^3(y)}{1+y^2}dy-3\int_0^1\frac{y\log^2(y)}{1+y^2}dy+6\int_0^1\frac{y\log(y)}{1+y^2}dy-6\int_0^1\frac{y}{1+y^2}dy\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{\frac{1}{16}\int_0^1\frac{\log^3(f)}{1+f}df-\frac{3}{8}\int_0^1\frac{\log^2(f)}{1+f}df+\frac{3}{2}\int_0^1\frac{f\log(f)}{1+f}df-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}$$

$$-\frac{1}{2}\left\{\frac{1}{16}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log^3(f)\,df-\frac{3}{8}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log^2(f)\,df\right.$$

$$\left.+\frac{3}{2}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log(f)\,df-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{3}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^4}-\frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}-\frac{3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{3}{8}\eta(4)-\frac{3}{4}\eta(3)-\frac{3}{2}\eta(2)-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{21}{64}\zeta(4)-\frac{9}{16}\zeta(3)-\frac{3}{4}\zeta(2)-3\log(2)\right\}$$

$$=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz$$

$$\frac{I}{3}=\int_0^1\int_0^1\int_0^1\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=\int_0^1\log(x)\int_0^1\left\{\tan^{-1}(xy)-\frac{\log(1+x^2y^2)}{2xy}\right\}dy\,dx$$

$$=\int_0^1\int_0^1\log(x)\tan^{-1}(xy)\,dy\,dx-\frac{1}{2}\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx$$

$$=\int_0^1\log(x)\tan^{-1}(x)\,dx-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x^2)}{x}dx$$

$$-\frac{1}{2}\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx=I_1-\frac{1}{2}I_2-\frac{1}{2}I_3$$

$$\int_0^1\log(x)\tan^{-1}(x)\,dx=-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{48}$$

$$\int_0^1\frac{\log(x)\log(1+x^2)}{x}dx=\frac{1}{4}\int_0^1\frac{\log(x)\log(1+x)}{x}dx=-\frac{3}{16}\zeta(3)$$

$$\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx=-\frac{1}{2}\int_0^1\frac{\log^2(x)\log(1+x^2)}{x}dx$$

$$=-\frac{1}{16}\int_0^1\frac{\log^2(x)\log(1+x)}{x}dx=-\frac{7\pi^4}{5760}$$

$$\frac{I}{3}=I_1-\frac{1}{2}I_2-\frac{1}{2}I_3=-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{48}+\frac{3}{32}\zeta(3)+\frac{7\pi^4}{11520}$$

$$I=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$

2309. Prove that

$$I=\int_0^\infty\frac{x\log^2(1+x)}{(1+x)(2+x)^3}dx=\frac{1}{6}\left(2\pi^2-9\zeta(3)-6\log(4)\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

Let x+1=y

$$I=\int_1^{\infty}\frac{(y-1)\log^2(y)}{y(1+y)^3}dy\Bigg]_{\frac{1}{y}=z}=\int_0^1\frac{z(1-z)\log^2(z)}{(1+z)^3}dz$$

$$=\int_0^1\frac{3(1+z)-(1+z)^2-2}{(1+z)^3}\log^2(z)\,dz$$

$$=3\int_0^1\frac{1}{(1+z)^2}\log^2(z)\,dz-\int_0^1\frac{1}{1+z}\log^2(z)\,dz-2\int_0^1\frac{1}{(1+z)^3}\log^2(z)\,dz$$

$$=3\int_0^1\left\{-\sum_{k=0}^{\infty}(-1)^k kz^{k-1}\right\}\log^2(z)\,dz-\int_0^1\left\{\sum_{k=0}^{\infty}(-1)^k z^k\right\}\log^2(z)\,dz$$

$$-2\int_0^1\left\{\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k k(k-1)z^{k-2}\right\}\log^2(z)\,dz$$

$$=-3\int_0^1\left\{\sum_{k=1}^{\infty}(-1)^k kz^{k-1}\right\}\log^2(z)\,dz-\int_0^1\left\{\sum_{k=0}^{\infty}(-1)^k z^k\right\}\log^2(z)\,dz$$

$$-\int_0^1\left\{\sum_{k=2}^{\infty}(-1)^k k(k-1)z^{k-2}\right\}\log^2(z)\,dz=$$

$$=-3\sum_{k=1}^{\infty}(-1)^k k\int_0^1 z^{k-1}\log^2(z)\,dz-\sum_{k=0}^{\infty}(-1)^k\int_0^1 z^k\log^2(z)\,dz$$

$$-\sum_{k=2}^{\infty}(-1)^k k(k-1)\int_0^1 z^{k-2}\log^2(z)\,dz$$

$$=-6\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}-2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}-2\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k-1)^2}$$

$$=-6\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}-2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}-2\sum_{k=2}^{\infty}\frac{(-1)^k}{k-1}-2\sum_{k=2}^{\infty}\frac{(-1)^k}{(k-1)^2}$$

$$=6\eta(2)-2\eta(3)-2\log(2)-2\eta(2)=4\eta(2)-2\eta(3)-2\log(2)$$

$$=2\zeta(2)-\frac{3}{2}\zeta(3)-2\log(2)=\frac{1}{6}\left(2\pi^2-9\zeta(3)-6\log(4)\right)$$

2310. If $a,b\in(0,1),n\in\mathbb{N}^*$ then:

$$\frac{(b-a)^2}{2}+\frac{(b-a)^3}{3}+\cdots+\frac{(b-a)^n}{n}+\int_a^b\frac{x^{n+1}}{1-x}dx\le\frac{b(b-a)}{1-b}$$

Proposed by Daniel Sitaru – Romania

Solution by Rajarshi Chakraborty-India

Let

$$f(x)=\frac{(b-x)^2}{2}+\frac{(b-x)^3}{3}\dots\frac{(b-x)^n}{n}+\int_x^b\frac{y^{n+1}}{1-y}dy-\frac{b(b-x)}{1-b}$$

$$\Rightarrow f(b)=0$$

$f(x)$ is clearly continuous in $(0,1)$

$$f'(x)=-(b-x)-(b-x)^2\dots-(b-x)^{n-1}-\frac{x^{n+1}}{1-x}+\frac{b}{1-b},x\in(0,b]$$

$$f'(b)=-\frac{b^{n+1}}{1-b}+\frac{b}{1-b}>0\text{ for }0<b<1$$

$$f'(0)=-b-b^2-b^3\dots-b^{n-1}+\frac{b}{1-b}>-\frac{b}{1-b}+\frac{b}{1-b}=0$$

Thus both $f'(b)$ and $f'(0)$ are positive

Now $f''(x)=1+2(b-x)+3(b-x)^2+\cdots(n-1)(b-x)^{n-2}$

$-\{(n+1)x^n+(n+2)x^{n+1}\dots\infty\}$

And $f'''(x)=-2x-6(b-x)\dots-(n-1)(n-2)(b-x)^{n-3}-$

$\{(n+1)nx^{n-1}+(n+2)(n+1)x^n\dots\infty\}<0$

$\Rightarrow f''(x)$ is a strictly decreasing function

$$f''(0)=1+2b+3b^2+\cdots(n-1)b^{n-2}>0$$

$$f''(b)=1-\{(n+1)b^n+(n+2)b^{n+1}\dots\infty\}$$

Case 1: If b is large enough then $f''(b)<0$

So $\exists c\in(0,b)$ s.t. $f''(c)=0$

Since $f''(x)$ is continuous, $f''(x)>0$ for $x\in(0,c)$

and $f''(x)<0$ for $x\in(c,b)$ which means

$f'(x)$ is increasing for $x\in(0,c)$

and $f'(x)$ is decreasing for $x\in(c,b)$

Now $f'(0) > 0$. It monotonically increases to a bigger positive value up to $x = c$

Then starts decreasing monotonically to a smaller positive value $f'(b) > 0$ at $x = b$

Thus $f'(x) > 0$ for $x \in (0, b)$

So $f(x)$ is an increasing function for $x \in (0, b)$

Case 2: if b is small enough $f''(b) > 0$

Thus $f'(x)$ is increasing for $x \in (0, b)$

And since $f'(0) > 0$ it follows that $f'(x) > 0$

Thus $f(x)$ is an increasing function for $x \in (0, b)$

So for $0 < a \le b < 1$

$$f(a) \le f(b)$$

$$\Rightarrow \frac{(b-a)^2}{2} + \frac{(b-a)^3}{3} \dots \frac{(b-a)^n}{n} + \int_a^b \frac{y^{n+1}}{1-y} dy - \frac{b(b-a)}{1-b} \le 0$$

$$\Rightarrow \frac{(b-a)^2}{2} + \frac{(b-a)^3}{3} \dots \frac{(b-a)^n}{n} + \int_a^b \frac{y^{n+1}}{1-y} dy \le \frac{b(b-a)}{1-b}$$

2311. Prove that

$$I = \int_0^\infty \int_0^1 \frac{\log(1+y)\log^2(1+x^2)}{x(1+y^2)} dx\, dy = \frac{G\zeta(3)}{8} + \frac{\pi}{32}\log(2)$$

where, $\zeta(3)$ is the *Apery's constant.*

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^\infty \int_0^1 \frac{\log(1+y)\log^2(1+x^2)}{x(1+y^2)} dx\, dy = \int_0^1 \frac{\log^2(1+x^2)}{x} dx * \int_0^\infty \frac{\log(1+y)}{1+y^2} dy = I_1 * I_2$$

$$I_1 = \int_0^1 \frac{\log^2(1+x^2)}{x} dx \Bigg]_{x^2=m} = \frac{1}{2}\int_0^1 \frac{\log^2(1+m)}{m} dm = \frac{1}{8}\zeta(3)$$

$$I_2=\int_0^\infty \frac{\log(1+y)}{1+y^2}dy=\int_0^1 \frac{\log(1+y)}{1+y^2}dy+\int_1^\infty \frac{\log(1+y)}{1+y^2}dy$$

$$=\int_0^1 \frac{\log(1+y)}{1+y^2}dy+\left\{\int_0^1 \frac{\log(1+y)}{1+y^2}dy-\int_0^1 \frac{\log(y)}{1+y^2}dy\right\}$$

$$=2\int_0^1 \frac{\log(1+y)}{1+y^2}dy-\int_0^1 \frac{\log(y)}{1+y^2}dy=\frac{\pi}{4}\log(2)+G$$

$$I=I_1*I_2=\frac{1}{8}\zeta(3)\left\{\frac{\pi}{4}\log(2)+G\right\}=\frac{G\zeta(3)}{8}+\frac{\pi}{32}\zeta(3)\log(2)$$

Note: $\int_0^1 \frac{\log^2(1+m)}{m}dm=\frac{1}{4}\zeta(3)$ and $\int_0^1 \frac{\log(1+y)}{1+y^2}dy=\frac{\pi}{8}\log(2)$

2312. Prove that:

$$\int_0^1 \frac{x^2\log(x)}{x^2+x+1}dx=-\frac{\pi^2}{54}-1+\frac{1}{9}\varphi^{(1)}\left(\frac{1}{3}\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1 \frac{x^2\log(x)}{x^2+x+1}dx=\int_0^1 \frac{(x^2-x^3)\log(x)}{1-x^3}dx=\int_0^1 \frac{x^2\log(x)}{1-x^3}dx-\int_0^1 \frac{x^3\log(x)}{1-x^3}dx$$

$$=\int_0^1 \frac{x^2\log(x)}{1-x^3}dx+\int_0^1 \log(x)\,dx-\int_0^1 \frac{\log(x)}{1-x^3}dx$$

$$=\frac{1}{9}\int_0^1 \frac{\log(y)}{1-y}dy-1-\frac{1}{9}\int_0^1 \frac{y^{-\frac{2}{3}}\log(y)}{1-y}dy=-\frac{\pi^2}{54}-1+\frac{1}{9}\varphi^{(1)}\left(\frac{1}{3}\right)$$

NOTE: $\int_0^1 \frac{x^{n-1}}{1-x}\log^m(x)\,dx=-\varphi^{(m)}(n)$

2313. Prove that

$$\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx=G+\frac{\pi}{4}\log(2)-\frac{\pi}{4}$$

where G denotes *Catalan's constant*

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$\int_0^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx = \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_1^\infty \frac{\tan^{-1}(x)}{x(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_0^1 \frac{x\tan^{-1}\left(\frac{1}{x}\right)}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x)}{x(1+x)^2}dx + \int_0^1 \frac{x\left(\frac{\pi}{2} - \tan^{-1}(x)\right)}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x) + \frac{\pi}{2}x^2 - x^2\tan^{-1}(x)}{x(1+x)^2}dx$$

$$= \int_0^1 \frac{(1-x^2)\tan^{-1}(x)}{x(1+x)^2}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= \int_0^1 \frac{(1-x)\tan^{-1}(x)}{x(1+x)}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\tan^{-1}(x)}{x}dx - 2\int_0^1 \frac{\tan^{-1}(x)}{1+x}dx + \frac{\pi}{2}\int_0^1 \frac{x}{(1+x)^2}dx$$

$$= G - \frac{\pi}{4}\log(2) + \frac{\pi}{2}\left(\log(2) - \frac{1}{2}\right) = G + \frac{\pi}{4}\log(2) - \frac{\pi}{4}$$

$$\text{Note: } \int_0^1 \frac{\tan^{-1}(x)}{x}dx = G \text{ and } \int_0^1 \frac{\tan^{-1}(x)}{1+x}dx = \frac{\pi}{8}\log(2)$$

2314. If $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\int_0^1 \left(1 - t^{\frac{1}{\sin^2 x}}\right)dt\right)\left(\int_0^1 \left(1 - t^{\frac{1}{\cos^2 x}}\right)dt\right) \le \int_0^1 (1-t)\,dt$$

Proposed by Daniel Sitaru – Romania

Solution by Samir HajAli-Damascus-Syria

First, let's consider a function f defined on $\left(0, \frac{\pi}{2}\right)$

$$f: x \to f(x) = \frac{1}{1+\cos^2 x} \cdot \frac{1}{1+\sin^2 x}$$

$$f'(x) = \frac{+2\cos x \sin x}{(1+\cos^2 x)^2} \cdot \frac{1}{1+\sin^2 x} + \frac{-2\sin x \cdot \cos x}{(1+\sin^2 x)^2} \cdot \frac{1}{1+\cos^2 x}$$

$$=\frac{\sin 2x}{(1+\cos^2 x)(1+\sin^2 x)}\left(\frac{1}{1+\cos^2 x}-\frac{1}{1+\sin^2 x}\right)$$

$$=\frac{\sin 2x}{(1+\cos^2 x)(1+\sin^2 x)}\left(\frac{\sin^2 x-\cos^2 x}{(1+\cos^2 x)(1+\sin^2 x)}\right)$$

Now:

$$f'(x)=0\Rightarrow \cos^2 x-\sin^2 x=0$$

$$\Rightarrow \cos(2x)=0\Rightarrow 2x=\frac{\pi}{2}+\pi k, k\in\mathbb{Z}\Rightarrow x=\frac{\pi}{4}+\frac{\pi k}{2}$$

For $k=0\Rightarrow x=\frac{\pi}{4}\Rightarrow f\left(\frac{\pi}{4}\right)=\frac{1}{1+\frac{1}{2}}\cdot\frac{1}{1+\frac{1}{2}}=\frac{4}{5}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'(x)$	$-\,-\,-\,-$	0	$+\,+\,+\,+\,+\,+$
$f(x)$	$\frac{1}{2}$ ↘	$\frac{4}{9}$ ↗	$\frac{1}{2}$

$$\Rightarrow \text{For } x\in\left(0,\frac{\pi}{2}\right), f(x)<\frac{1}{2}$$

$$\left(\int_0^1\left(1-t^{\frac{1}{\sin^2 x}}\right)dt\right)\left(\int_0^1\left(1-t^{\frac{1}{\cos^2 x}}\right)dt\right)\le\int_0^1(1-t)\,dt$$

$$\left(1-\frac{t^{\frac{1}{\sin^2 x}+1}}{\frac{t}{\sin^2 x}+1}\Bigg|_0^1\right)\left(1-\frac{t^{\frac{1}{\cos^2 x}+1}}{\frac{1}{\cos^2 x}+1}\Bigg|_0^1\right)\le 1-\frac{t^2}{2}\Bigg|_0^1$$

$$\Leftrightarrow\left(1-\frac{1}{\frac{1}{\sin^2 x}+1}\right)\left(1-\frac{1}{\frac{1}{\cos^2 x}+1}\right)\le 1-\frac{1}{2}$$

$$\Leftrightarrow\left(1-\frac{\sin^2 x}{1+\sin^2 x}\right)\left(1-\frac{\cos^2 x}{1+\cos^2 x}\right)\le\frac{1}{2}\Leftrightarrow\left(\frac{1}{1+\sin^2 x}\right)\left(\frac{1}{1+\cos^2 x}\right)\le\frac{1}{2}$$

$$\Leftrightarrow f(x)\le\frac{1}{2}\Leftrightarrow x\in\left(0,\frac{\pi}{2}\right)$$

2315. **Prove the below closed form**

$$I=\int_0^\infty \frac{J_{1/2}(x)J_{-1/2}(x)}{1+x^2}dx=\frac{e^2-1}{2e^2}$$

Where, $J_n(x)$ is the Bessel function of the first kind.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty \frac{J_{1/2}(x)J_{-1/2}(x)}{1+x^2}dx=\int_0^\infty \frac{\sqrt{\frac{2}{\pi x}}\sin(x)\sqrt{\frac{2}{\pi x}}\cos(x)}{1+x^2}dx=\frac{1}{\pi}\int_0^\infty \frac{\sin(2x)}{x(1+x^2)}dx$$

$$=\frac{1}{\pi}\int_0^\infty \frac{\sin(2x)}{x}dx-\frac{1}{\pi}\int_0^\infty \frac{x\sin(2x)}{1+x^2}dx=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{x\sin(2x)}{1+x^2}dx$$

$$\begin{cases} f(z)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\cos(xt)dt\right)\cos(zx)dx \\ f(z)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\sin(xt)dt\right)\sin(zx)dx \end{cases},\text{put } f(z)=e^{-z}$$

$$\Rightarrow\begin{cases} e^{-z}=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty e^{-t}\cos(xt)dt\right)\cos(zx)\,dx \\ e^{-z}=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty e^{-t}\sin(xt)dt\right)\sin(zx)\,dx \end{cases}$$

$$\Rightarrow\begin{cases} e^{-z}=\frac{2}{\pi}\int_0^\infty \frac{\cos(zx)}{1+x^2}dx \\ e^{-z}=\frac{2}{\pi}\int_0^\infty \frac{x\sin(zx)}{1+x^2}dx \end{cases}\Rightarrow\begin{cases} \frac{\pi}{2}e^{-z}=\int_0^\infty \frac{\cos(zx)}{1+x^2}dx \\ \frac{\pi}{2}e^{-z}=\int_0^\infty \frac{x\sin(zx)}{1+x^2}dx \end{cases}\Rightarrow\int_0^\infty \frac{x\sin(2x)}{1+x^2}dx$$

$$=\frac{\pi}{2}e^{-2}$$

$$I=\frac{1}{2}-\frac{1}{\pi}\left(\frac{\pi}{2}e^{-2}\right)=\frac{1}{2}-\frac{1}{2e^2}=\frac{e^2-1}{2e^2}$$

2316. Prove the below closed form

$$I = \int_0^1 \int_0^1 \int_0^1 \sum_{x,y,z} \left(\frac{\sqrt{xy} + \sqrt{yz}}{\sqrt{yz} + \sqrt{zx}} \right) dx\, dy\, dz = \frac{48}{5}\log(2) - \frac{33}{10}$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^1 \int_0^1 \int_0^1 \sum_{x,y,z} \left(\frac{\sqrt{xy} + \sqrt{yz}}{\sqrt{yz} + \sqrt{zx}} \right) dx\, dy\, dz = 3 \int_0^1 \int_0^1 \int_0^1 \frac{\sqrt{y}(\sqrt{x} + \sqrt{z})}{\sqrt{z}(\sqrt{x} + \sqrt{y})} dx\, dy\, dz$$

$$\text{Let } \sqrt{x} \to m, \sqrt{y} \to n, \sqrt{z} \to p$$

$$I = 24 \int_0^1 \int_0^1 \int_0^1 \frac{mn^2(m+p)}{m+n} dp\, dn\, dm = 24 \int_0^1 \int_0^1 \frac{m\left(m + \frac{1}{2}\right)n^2}{m+n} dn\, dm$$

$$= 24 \int_0^1 m^3 \left(m + \frac{1}{2}\right) \log(1+m)\, dm + 12 \int_0^1 m\left(m + \frac{1}{2}\right) dm$$

$$- 24 \int_0^1 m^2 \left(m + \frac{1}{2}\right) dm - 24 \int_0^1 m^3 \left(m + \frac{1}{2}\right) \log(m)\, dm$$

$$= 24\left(\frac{320\log(2) - 67}{800}\right) + 12\left(\frac{7}{12}\right) - 24\left(\frac{5}{12}\right) - 24\left(-\frac{57}{800}\right)$$

$$= \frac{48}{5}\log(2) - \frac{33}{10}$$

2317. If $\alpha, \beta, \gamma \geq 0$, then :

$$3 + \frac{(\alpha + \beta + \gamma)^4}{135} \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\sum_{\substack{cyc \\ \alpha,\beta,\gamma}} \sqrt[n^2]{e^{(k\alpha)^2}} \right)$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(m) = e^m - 1 - m - \frac{m^2}{2}\ \forall\, m \geq 0 \text{ and then} : f'(m) = e^m - 1 - m \geq 0$$

$$\Rightarrow f(m) \text{ is} \uparrow \text{on } [0, \infty) \Rightarrow f(m) \geq f(0) = 1 - 1 \therefore e^m \geq 1 + m + \frac{m^2}{2}\ \forall\, m \geq 0 \to (1)$$

$$\text{Now, } n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1 \text{ and putting}$$

$$n = 1, 2, 3 \ldots, (n-1), n \text{ successively, we arrive at} :$$

$$1^5 - 0^5 = 5.1^4 - 10.1^3 + 10.1^2 - 5.1 + 1$$

$$2^5-1^5=5.2^4-10.2^3+10.2^2-5.2+1$$

$$\dots\dots$$

$$(n-1)^5-(n-2)^5=5.(n-1)^4-10.(n-1)^3+10.(n-1)^2-5.(n-1)+1$$

$$n^5-(n-1)^5=5.n^4-10.n^3+10.n^2-5.n+1;\ \textit{and summing up, we arrive at}:$$

$$n^5=5\sum_{k=1}^{n}k^4-10\sum_{k=1}^{n}k^3+10\sum_{k=1}^{n}k^2-5\sum_{k=1}^{n}k+n$$

$$=5\sum_{k=1}^{n}k^4-10.\frac{n^2(n+1)^2}{4}+10.\frac{n(n+1)(2n+1)}{6}-5.\frac{n(n+1)}{2}+n$$

$$\Rightarrow\sum_{k=1}^{n}k^4=\frac{6n^5+15n^4+10n^3-n}{30}\to(2)$$

$$\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)=\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}e^{\frac{k^2\alpha^2}{n^2}}\right)\overset{via\ (1)}{\geq}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\left(1+\frac{k^2\alpha^2}{n^2}+\frac{k^4\alpha^4}{n^4}\right)\right)$$

$$=\sum_{k=1}^{n}\left(3+\frac{k^2}{n^2}\left(\sum_{cyc}\alpha^2\right)+\frac{k^4}{n^4}\left(\sum_{cyc}\alpha^4\right)\right)=3n+\frac{\sum_{cyc}\alpha^2}{n^2}.\sum_{k=1}^{n}k^2+\frac{\sum_{cyc}\alpha^4}{n^4}.\sum_{k=1}^{n}k^4$$

$$\overset{via\ (2)}{=}3n+\frac{\sum_{cyc}\alpha^2}{n^2}.\frac{n(n+1)(2n+1)}{6}+\frac{\sum_{cyc}\alpha^4}{n^4}.\frac{6n^5+15n^4+10n^3-n}{30}$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)$$

$$=\lim_{n\to\infty}\left(3+\frac{\sum_{cyc}\alpha^2}{6}.\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{\sum_{cyc}\alpha^4}{30}.\left(6+\frac{15}{n}+\frac{10}{n^2}-\frac{1}{n^4}\right)\right)$$

$$=3+\frac{\sum_{cyc}\alpha^2}{3}+\frac{\sum_{cyc}\alpha^4}{5}\overset{Holder}{\geq}3+\frac{\sum_{cyc}\alpha^2}{3}+\frac{(\alpha+\beta+\gamma)^4}{5.27}\overset{\alpha,\beta,\gamma\geq0}{\geq}3+\frac{(\alpha+\beta+\gamma)^4}{135}$$

$$\therefore 3+\frac{(\alpha+\beta+\gamma)^4}{135}\leq\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{cyc\\ \alpha,\beta,\gamma}}\sqrt[n^2]{e^{(k\alpha)^2}}\right)\ \forall\ \alpha,\beta,\gamma\geq0\ (QED)$$

2318. Prove the below closed form

$$I=\int_0^1\int_0^1\int_0^1\frac{\ln(x+y+z)}{x+y+z}dx\,dy\,dz=9\ln(2)-\frac{27}{4}\ln(3)-3\ln^2(2)+\frac{9}{4}\ln^2(3)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\text{Let } x+y+z\to m$$

$$I=\int_0^1\int_0^1\int_{x+y}^{x+y+1}\frac{\ln(m)}{m}dm\,dy\,dx \overset{IBP}{=} \int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx-\int_0^1\int_0^1\frac{y\ln(x+y+1)}{x+y+1}dy\,dx\Bigg]_{x+y+1=m}$$

$$+\int_0^1\int_0^1\frac{y\ln(x+y)}{x+y}dy\,dx\Bigg]_{x+y=m}$$

$$=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx-\int_0^1\int_{x+1}^{x+2}\frac{(m-1-x)\ln(m)}{m}dm\,dx$$

$$+\int_0^1\int_x^{x+1}\frac{(m-x)\ln(m)}{m}dm\,dx$$

$$=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx-\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx+\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx+\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx$$

$$+\int_0^1\int_x^{x+1}\ln(m)\,dm\,dx-\int_0^1 x\int_x^{x+1}\frac{\ln(m)}{m}dm\,dx$$

$$=2\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx-\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx+\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx+\int_0^1\int_x^{x+1}\ln(m)\,dm\,dx$$

$$-\int_0^1 x\int_x^{x+1}\frac{\ln(m)}{m}dm\,dx$$

$$I_1=\int_0^1\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx \overset{IBP}{=} \int_2^3\frac{\ln(m)}{m}dm-\int_0^1\frac{x\ln(x+2)}{x+2}dx+\int_0^1\frac{x\ln(x+1)}{x+1}dx$$

$$=\frac{3}{2}\ln^2(3)-2\ln^2(2)-3\ln(3)+4\ln(2)$$

$$I_2=\int_0^1\int_{x+1}^{x+2}\ln(m)\,dm\,dx \overset{IBP}{=} \int_2^3\ln(m)\,dm-\int_0^1 x\ln(x+2)\,dx+\int_0^1 x\ln(x+1)\,dx$$

$$=\frac{9}{2}\ln(3)-4\ln(2)-\frac{3}{2}$$

$$I_3=\int_0^1 x\int_{x+1}^{x+2}\frac{\ln(m)}{m}dm\,dx \overset{IBP}{=} \frac{1}{2}\int_2^3\frac{\ln(m)}{m}dm-\frac{1}{2}\int_0^1\frac{x^2\ln(x+2)}{x+2}dx+\frac{1}{2}\int_0^1\frac{x^2\ln(x+1)}{x+1}dx$$

$$=-\frac{3}{4}\ln^2(3)+\ln^2(2)+\frac{15}{4}\ln(3)-4\ln(2)-\frac{3}{4}$$

$$I_4=\int_0^1\int_x^{x+1}\ln(m)\,dm\,dx \overset{IBP}{=} \int_1^2\ln(m)\,dm-\int_0^1 x\ln(x+1)\,dx+\int_0^1 x\ln(x)\,dx=2\ln(2)-\frac{3}{2}$$

$$I_5=\int_0^1 x\int_x^{x+1}\frac{\ln(m)}{m}dm\,dx \overset{IBP}{=} \frac{1}{2}\int_1^2\frac{\ln(m)}{m}dm-\frac{1}{2}\int_0^1\frac{x^2\ln(x+1)}{x+1}dx+\frac{1}{2}\int_0^1\frac{x^2\ln(x)}{x}dx$$

$$=\ln(2)-\frac{3}{4}$$

$$I=2I_1-I_2+I_3+I_4-I_5=9\ln(2)-\frac{27}{4}\ln(3)-3\ln^2(2)+\frac{9}{4}\ln^2(3)$$

2319. Prove the below closed form

$$I=\int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+yx)(1+xyz)}dx\,dy\,dz=\frac{7}{8}\varsigma(3)-\frac{\pi^2}{12}\log(2)+\frac{1}{6}\log^3(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+yx)(1+xyz)}dz\,dy\,dx=\int_0^1\int_0^1\frac{\log(1+xy)}{xy(1+x)(1+yx)}dy\,dx$$

$$\text{Let } xyz\to m$$

$$I=\int_0^1\frac{1}{x(1+x)}\int_0^x\frac{\log(1+m)}{m(1+m)}dm\,dx \overset{IBP}{=}\left[(\log(x)-\log(1+x))\int_0^x\frac{\log(1+m)}{m(1+m)}dm\right]_0^1$$

$$-\int_0^1\frac{\log(x)\log(1+x)-\log^2(1+x)}{x(1+x)}dx$$

$$=-\log(2)\int_0^1\frac{\log(1+m)}{m(1+m)}dm-\int_0^1\frac{\log(x)\log(1+x)}{x(1+x)}dx+\int_0^1\frac{\log^2(1+x)}{x(1+x)}dx$$

$$I_1=\int_0^1\frac{\log(1+m)}{m(1+m)}dm=\int_0^1\frac{\log(1+m)}{m}dm-\int_0^1\frac{\log(1+m)}{1+m}dm=\frac{\pi^2}{12}-\frac{1}{2}\log^2(2)$$

$$I_2=\int_0^1\frac{\log(x)\log(1+x)}{x(1+x)}dx=\int_0^1\frac{\log(x)\log(1+x)}{x}dx-\int_0^1\frac{\log(x)\log(1+x)}{1+x}dx$$

$$=-\frac{3}{4}\varsigma(3)+\frac{1}{8}\varsigma(3)=-\frac{5}{8}\varsigma(3)$$

$$I_3=\int_0^1\frac{\log^2(1+x)}{x(1+x)}dx=\int_0^1\frac{\log^2(1+x)}{x}dx-\int_0^1\frac{\log^2(1+x)}{1+x}dx=\frac{1}{4}\varsigma(3)-\frac{1}{3}\log^3(2)$$

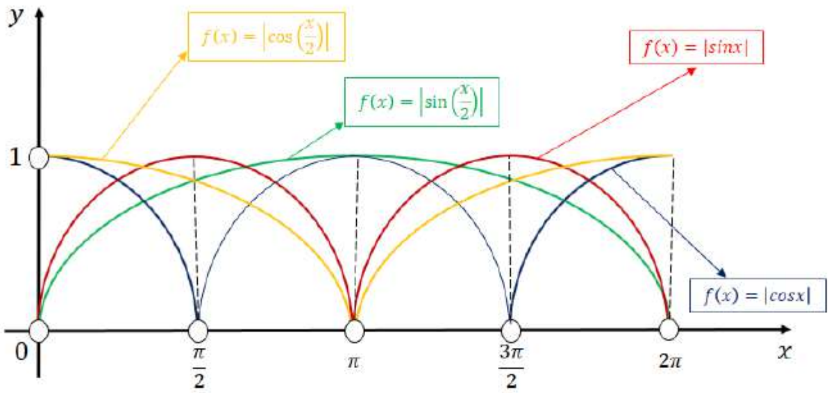
$$I=-\log(2)\,I_1-I_2+I_3=\frac{7}{8}\varsigma(3)-\frac{\pi^2}{12}\log(2)+\frac{1}{6}\log^3(2)$$

2320. Prove that

$$I=\int_0^{2\pi}\left(|\sin x|+|\cos x|+\left|\sin\left(\frac{x}{2}\right)\right|+\left|\cos\left(\frac{x}{2}\right)\right|\right)dx=16$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Togrul Ehmedov-Azerbaijan



$$I_1=\int_0^{2\pi}|\sin x|\,dx=2\int_0^{\pi}\sin x\,dx=4$$

$$I_2=\int_0^{2\pi}|\cos x|\,dx=4\int_0^{\frac{\pi}{2}}\cos x\,dx=4$$

$$I_3=\int_0^{2\pi}\left|\sin\left(\frac{x}{2}\right)\right|dx=\int_0^{2\pi}\sin\left(\frac{x}{2}\right)dx=4$$

$$I_4=\int_0^{2\pi}\left|\cos\left(\frac{x}{2}\right)\right|dx=2\int_0^{\pi}\cos\left(\frac{x}{2}\right)dx=4$$

$$I=I_1+I_2+I_3+I_4=16$$

2321.

Prove that

$$I=\int_1^{\infty}\frac{\log^2(x)}{(x+1)(x+2)}dx=\frac{1}{12}\left(-6\mathrm{Li}_3\left(\frac{1}{4}\right)+3\zeta(3)+8\log^3(2)+\pi^2\log(4)\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_1^\infty \frac{\log^2(x)}{(x+1)(x+2)}dx=\int_0^1 \frac{\log^2(x)}{(x+1)(2x+1)}dx=2\int_0^1 \frac{\log^2(x)}{1+2x}dx-\int_0^1 \frac{\log^2(x)}{1+x}dx$$

$$=2\sum_{k=0}^\infty (-1)^k 2^k \int_0^1 x^k\log^2(x)\,dx-\sum_{k=0}^\infty (-1)^k \int_0^1 x^k\log^2(x)\,dx$$

$$=-2\sum_{k=0}^\infty \frac{(-2)^{k+1}}{(k+1)^3}-2\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^3}=2\mathrm{Li}_3(-2)-2\eta(3)$$

$$=-2\left\{-\frac{1}{3}\log^3(2)-\frac{\pi^2}{12}\log(2)+\frac{1}{4}\mathrm{Li}_3\left(\frac{1}{4}\right)-\frac{7}{8}\varsigma(3)\right\}-\frac{3}{2}\varsigma(3)$$

$$=-\frac{1}{2}\mathrm{Li}_3\left(\frac{1}{4}\right)+\frac{\pi^2}{6}\log(2)+\frac{2}{3}\log^3(2)+\frac{1}{4}\varsigma(3)$$

$$=\frac{1}{12}\left(-6\mathrm{Li}_3\left(\frac{1}{4}\right)+3\varsigma(3)+8\log^3(2)+\pi^2\log(4)\right)$$

NOTE: $$\begin{cases}\mathrm{Li}_3(z)+\mathrm{Li}_3(-z)=\frac{1}{4}\mathrm{Li}_3(z^2)\\ \mathrm{Li}_3(-z)-\mathrm{Li}_3\left(-\frac{1}{z}\right)=-\frac{1}{6}\log^3(z)-\frac{\pi^2}{6}\log(z)\end{cases}$$

$$\mathrm{Li}_3(-2)=-\frac{1}{3}\log^3(2)-\frac{\pi^2}{12}\log(2)+\frac{1}{4}\mathrm{Li}_3\left(\frac{1}{4}\right)-\frac{7}{8}\varsigma(3)$$

2322. Prove that

$$I=\int_0^1 x(\log(\log x)+\arctan(x-1))\,dx=\frac{1}{2}(i\pi-\gamma-1)$$

γ is the *Euler-Mascheroni constant.*

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1 x(\log(\log x)+\arctan(x-1))\,dx=\int_0^1 x\log(\log(x))\,dx+\int_0^1 x\arctan(x-1)\,dx$$

$$I_1=\int_0^1 x\log(\log(x))\,dx\Big]_{\log(x)=-y}=\int_0^\infty e^{-2y}\log(-y)\,dy$$

$$=\log(-1)\int_0^\infty e^{-2y}\,dy+\int_0^\infty e^{-2y}\log(y)\,dy\Big]_{2y=z}$$

$$=\frac{i\pi}{2}+\frac{1}{2}\int_0^\infty e^{-z}\log(z)\,dz-\frac{\log(2)}{2}\int_0^\infty e^{-z}\,dz=\frac{i\pi}{2}-\frac{1}{2}\gamma-\frac{\log(2)}{2}$$

$$=\frac{1}{2}(i\pi-\gamma-\log(2))$$

$$I_2=\int_0^1 x\arctan(x-1)\,dx=-\int_0^1(1-x)\arctan(x)\,dx=\frac{1}{2}(\log(2)-1)$$

$$I=I_1+I_2=\frac{1}{2}(i\pi-\gamma-1),\qquad \text{Note: }\int_0^\infty e^{-z}\log(z)\,dz=-\gamma$$

2323. Find a closed form:

$$I=\int_0^\infty \frac{\log(x)}{(1+x)(1+x^2)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty \frac{\log(x)}{(1+x)(1+x^2)}dx=\int_0^1 \frac{\log(x)}{(1+x)(1+x^2)}dx+\int_1^\infty \frac{\log(x)}{(1+x)(1+x^2)}dx\Big]_{x\to\frac{1}{x}}$$

$$=\int_0^1 \frac{\log(x)}{(1+x)(1+x^2)}dx-\int_0^1 \frac{x\log(x)}{(1+x)(1+x^2)}dx$$

$$=2\int_0^1 \frac{\log(x)}{(1+x)(1+x^2)}dx-\int_0^1 \frac{\log(x)}{1+x^2}dx$$

$$=\int_0^1 \frac{\log(x)}{1+x}dx-\int_0^1 \frac{x\log(x)}{1+x^2}dx\Big]_{x\to\frac{1}{x}}=\int_0^1 \frac{\log(x)}{1+x}dx-\frac{1}{4}\int_0^1 \frac{\log(x)}{1+x}dx$$

$$=\frac{3}{4}\int_0^1 \frac{\log(x)}{1+x}dx=\frac{3}{4}\sum_{k=0}^\infty(-1)^k\int_0^1 x^k\log(x)\,dx=-\frac{3}{4}\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2}$$

$$=-\frac{3}{4}\eta(2)=-\frac{3}{8}\varsigma(2)=-\frac{\pi^2}{16}$$

2324. Find:

$$I = \int \left(\frac{\cos\left(\frac{\pi}{3} - x\right) \sin(3x) \cos\left(\frac{\pi}{3} + x\right)}{\sin\left(\frac{\pi}{3} - x\right) \cos(3x) \sin\left(\frac{\pi}{3} + x\right)} \right)^2 dx$$

Proposed by Daniel Sitaru-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int \left(\frac{\cos\left(\frac{\pi}{3} - x\right) \sin(3x) \cos\left(\frac{\pi}{3} + x\right)}{\sin\left(\frac{\pi}{3} - x\right) \cos(3x) \sin\left(\frac{\pi}{3} + x\right)} \right)^2 dx$$

$$= \int \left(\frac{\cos\left(\frac{\pi}{3} - x\right) \cos(x) \cos\left(\frac{\pi}{3} + x\right) \sin(3x)}{\sin\left(\frac{\pi}{3} - x\right) \sin(x) \sin\left(\frac{\pi}{3} + x\right) \cos(3x)} \right)^2 \left(\frac{\sin(x)}{\cos(x)}\right)^2 dx$$

$$= \int \left(\frac{\tan(3x)}{\tan\left(\frac{\pi}{3} - x\right) \tan(x) \tan\left(\frac{\pi}{3} + x\right)} \right)^2 (\tan(x))^2 dx$$

We know that $\tan\left(\frac{\pi}{3} - x\right) \tan(x) \tan\left(\frac{\pi}{3} + x\right) = \tan(3x)$

Then we can write

$$I = \int \left(\frac{\tan(3x)}{\tan(3x)}\right)^2 (\tan(x))^2 dx = \int (\tan(x))^2 dx = \tan(x) - x + C$$

2325. Find:

$$\Omega = \int x^{\log_2 x} \cdot (1 + 2\log_2 x)\, dx$$

Proposed by Daniel Sitaru-Romania

Solution by Igor Soposki-Skopje-Macedonia

$$u = x^{\log_2 x + 1}, \qquad \log_2 u = \log_2 x^{\log_2 x + 1}$$

$$\log_2 u = (\log_2 x + 1)\log_2 x, \qquad \log_2 u = (\log_2 x)^2 + \log_2 x$$

$$\frac{du}{u\ln 2} = \left(2\log_2 x \cdot \frac{1}{x\ln 2} + \frac{1}{x\ln 2}\right) dx, \qquad \frac{du}{u} = \left(2\log_2 x \cdot \frac{1}{x} + \frac{1}{x}\right) dx$$

$$du = \frac{u(1 + 2\log_2 x)}{x} dx, \qquad du = \frac{x^{\log_2 x + 1}(1 + 2\log_2 x)}{x} dx$$

$$du = x^{\log_2 x} \cdot (1 + 2\log_2 x) dx$$

$$\Omega=\int x^{log_2 x}\cdot(1+2log_2 x)\,dx=\int du=u+C=x^{log_2 x+1}+C$$

2326. Prove that

$$\int_0^{\infty}\frac{dx}{x^4+x^2+1}=\frac{\pi}{2\sqrt{3}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\infty}\frac{dx}{x^4+2x^2\cos(2a)+1}\Bigg]_{a=\frac{\pi}{6}\text{ and }x=\frac{1}{t}}=\int_0^{\infty}\frac{t^2dt}{t^4+2t^2\cos(2a)+1}$$

$$2I=\int_0^{\infty}\frac{dx}{x^4+2x^2\cos(2a)+1}+\int_0^{\infty}\frac{x^2dx}{x^4+2x^2\cos(2a)+1}=\int_0^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}$$

$$I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2(1-2\sin^2(a))+1}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{x^4+2x^2+1-4x^2\sin^2(a)}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1)^2-4x^2\sin^2(a)}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

We know that $A=\frac{1}{4}\int_{-\infty}^{\infty}\frac{-2x\sin(a)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}=0$

Then let's clarify the sum of A+I

$$A+I=I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$+\frac{1}{4}\int_{-\infty}^{\infty}\frac{-2x\sin(a)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(x^2-2x\sin(a)+1)dx}{(x^2+1-2x\sin(a))(x^2+1+2x\sin(a))}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{x^2+1+2x\sin(a)}$$

$$I=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{x^2+\sin^2(a)+\cos^2(a)+2x\sin(a)}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{(x+\sin(a))^2+\cos^2(a)}$$

$$=\frac{1}{4\cos(a)}\text{arctg}\left(\frac{x+\sin(a)}{\cos(a)}\right)\Bigg]_{-\infty}^{\infty}=\frac{\pi}{4\cos(a)}\Bigg]_{a=\frac{\pi}{6}}=\frac{\pi}{2\sqrt{3}}$$

2327. Prove the below closed form

$$I=\int_0^1 \frac{x\log(x)}{(x+1)(x^2+1)}dx=\frac{1}{32}(\pi^2-16G)$$

Where, G is Catalan's constant

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$u=\log(x)\to du=dx/x$$

$$dv=\frac{x\log(x)}{(x+1)(x^2+1)}dx\to v=\frac{\tan^{-1}(x)}{2}-\frac{\log(x+1)}{2}+\frac{\log(x^2+1)}{4}$$

$$I\overset{IBP}{=}-\frac{1}{2}\int_0^1\frac{\tan^{-1}(x)}{x}dx+\frac{1}{2}\int_0^1\frac{\log(x+1)}{x}dx-\frac{1}{4}\int_0^1\frac{\log(x^2+1)}{x}dx\Bigg]_{x^2\to x}$$

$$=-\frac{1}{2}\int_0^1\frac{\tan^{-1}(x)}{x}dx+\frac{1}{2}\int_0^1\frac{\log(x+1)}{x}dx-\frac{1}{8}\int_0^1\frac{\log(x+1)}{x}dx=$$

$$=-\frac{1}{2}\int_0^1\frac{\tan^{-1}(x)}{x}dx+\frac{3}{8}\int_0^1\frac{\log(x+1)}{x}dx$$

$$=-\frac{1}{2}G+\frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}dx=-\frac{1}{2}G+\frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$

$$=-\frac{1}{2}G+\frac{3}{8}\eta(2)=-\frac{1}{2}G+\frac{3}{16}\varsigma(2)=-\frac{1}{2}G+\frac{\pi^2}{32}$$

2328. Find:

$$\int_0^1\int_0^1 (x\log(arccos(1-y))+arctan^2(1-y))\,dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1 x\log(\cos^{-1}(1-y))\,dxdy+\int_0^1\int_0^1 arctan^2(1-y)\,dxdy=$$
$$\frac{1}{2}\int_0^1\log(\cos^{-1}(y))\,dy$$

$$\text{Sub...}\left\{\arccos(y)=t;\ \frac{dt}{dy}=-\frac{1}{\sin(t)};\ t\left[0;\frac{\pi}{2}\right]\right\}$$

$$\Omega_1=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(t)\log(t)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos(t)}{t}dt+\frac{1}{2}\log(0)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos(t)-1}{t}dt+\frac{1}{2}\ln\left(\frac{\pi}{2}\right)$$

$$\left\{We\ know\ cosine\ integral\ Ci(z)=\gamma+\ln(z)+\int_0^z\frac{\cos(t)-1}{t}dz=-\int_0^\infty\frac{\cos(t)}{t}dt\right\}$$

$$\Omega_1=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{\cos(t)-1}{t}dt+\ln\left(\frac{\pi}{2}\right)\right)=\frac{1}{2}Ci\left(\frac{\pi}{2}\right)-\frac{\gamma}{2}$$

$$\Omega_2=\int_0^1\int_0^1 arctan^2(1-y)\,dxdy=\int_0^1 arctan^2(y)\,dy=\Big|_0^1 yarctan^2(y)-$$

$$-2\int_0^1\frac{yarctn(y)}{1+y^2}dy=|\arctan(y)=t|=\frac{\pi^2}{16}-2\int_0^{\frac{\pi}{4}}t.tan(t)dt=$$

$$=\frac{\pi^2}{16}+2\Big|_0^{\frac{\pi}{4}}t\log(\cos(t))-2\int_0^{\frac{\pi}{4}}\log(\cos(t))dt=C$$

$$\left\{We\ know\ Fourier\ series\ of.\ \ln(\cos(z))\right.$$

$$\left.=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nz)}{n}\ \ function\right\}$$

$$\Omega_2=\frac{\pi^2}{16}-\frac{\pi}{4}\ln(2)+\ln(2)\int_0^{\frac{\pi}{4}}dt$$

$$+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nt)\,dt$$

$$=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)++\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{e^2}=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)-G$$

2329. Find:

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\log(\sin(y)+\cos(y))dy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\log(\sin(y)+\cos(y))dy\quad\leftrightarrow\sin(y)+\cos(y)=\sqrt{2}\sin(y+\frac{\pi}{4})$$

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sqrt{2}\sin(y+\tfrac{\pi}{4})\right)dy=\tfrac{1}{2}\ln(2)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ydy+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sin\left(x+\tfrac{\pi}{4}\right)\right)dy$$

$$\Omega_1=\frac{\ln(2)}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ydy=\frac{\ln(2)}{2}\frac{y^2}{2}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{3\pi^2}{64}\ln(2)$$

$$\Omega_2=\frac{\ln(2)}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\sin\left(x+\frac{\pi}{4}\right)\right)dy=\left|y+\frac{\pi}{4}=x\right|=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\left(x-\frac{\pi}{4}\right)\ln(x)dx=I_1-I_2$$

$$I_1=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\ln(\sin x)dx=-\ln(2)\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} dx-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\cos(2nx)dx=$$

$$=-\frac{5\pi^2\ln(2)}{32}-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\cos(2nx)}{4n^2}-\frac{x\sin(2nx)}{2n}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}=-\frac{5\pi^2\ln(2)}{32}-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{3\pi\sin\left(\frac{3\pi n}{2}\right)}{8n^2}-\frac{x\sin(\pi n)}{4n^2}-\right.$$

$$\left.-\frac{\cos(\pi n)}{4n^3}+\frac{\cos\left(\frac{3\pi n}{2}\right)}{4n^3}\right]=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-$$

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{3\pi n}{2}\right)}{n^3}=$$

$$=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\frac{1}{4}(1-2^{1-3})\zeta(3)$$

$$+\frac{1}{4}\left(-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{6^3}+\frac{1}{8^3}-\cdots\right)=$$

$$=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}G-\frac{3}{4}\zeta(3)-\frac{1}{4}\times\frac{1}{2^3}\left(1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\cdots\right)=$$

$$=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}G-\frac{3}{16}\zeta(3)+\frac{1}{32}(1-2^{1-3})\zeta(3)$$

$$=-\frac{5\pi^2\ln(2)}{32}+\frac{3\pi}{8}G-\frac{21}{128}\zeta(3)$$

$$I_2=\frac{\pi}{4}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\ln(\sin x)dx=-\frac{\pi}{4}\ln(2)\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} dx-\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\cos(2nx)dx=-\frac{\pi^2\ln(2)}{16}$$

$$--\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\sin\left(\frac{3\pi n}{2}\right)-\sin(\pi n)}{2n}\right]$$

$$=-\frac{\pi^2\ln(2)}{16}-\frac{\pi}{8}\sum_{n=}^{\infty}\frac{\sin\left(\frac{3\pi n}{2}\right)}{n^2}+\frac{\pi}{8}\sum_{n=}^{\infty}\frac{\sin(\pi n)}{n^2}=\frac{\pi}{8}G-\frac{\pi^2\ln(2)}{16}$$

$$\Omega_2 = I_1 - I_2 = \frac{\pi}{4}G - \frac{3\pi^2\ln(2)}{32} - \frac{21}{128}\zeta(3) \qquad \Omega = \Omega_1 + \Omega_2 = \frac{\pi}{4}G - \frac{3\pi^2\ln(2)}{64} - \frac{21}{128}\zeta(3)$$

2330. Prove that:

$$\Omega = \int_0^\infty \frac{x\ln^2(1+x)}{(1+x)(2+x)(3+x)}dx = |1+x=t| = \int_1^\infty \frac{(t-1)\ln^2(t)}{t(t+1)(t+2)}dt$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\text{Replace } \left\{\frac{1}{t} = u \quad \frac{du}{dt} = -u^2 \quad u[0;1]\right\}$$

$$\Omega = \int_0^1 \frac{\left(\frac{1}{u}-1\right)\ln^2(u)}{\frac{1}{u}\left(\frac{1}{u}+1\right)\left(\frac{1}{u}+2\right)}\frac{du}{u^2} = \int_0^1 \frac{(1-u)\ln^2(u)}{(1+u)(1+2u)}du = 2\int_0^1 \frac{(1-u)\ln^2(u)}{(1+2u)}du - \int_0^1 \frac{(1-u)\ln^2(u)}{(1+u)}du =$$

$$= 2\Omega_1 - \Omega_2$$

$$\Omega_1 = \int_0^1 \frac{\ln^2(u)}{(1+2u)}du - \int_0^1 \frac{u\ln^2(u)}{(1+2u)}du = \frac{1}{2}\left(\int_0^1 \frac{\ln^2(u)}{\left(u+\frac{1}{2}\right)}du - \int_0^1 \frac{u\ln^2(u)}{\left(u+\frac{1}{2}\right)}du\right)$$

$$\text{General solution: } I(a) = \int \frac{\ln^2(x)}{x+a}dx \qquad \text{Replace: } \left\{x = at \quad \frac{dx}{dt} = a\right\}$$

$$\circledast I(a) = \int \frac{\ln^2(at)}{1+t}dt = \ln^2(at)\ln(1+t) - 2\int \frac{\ln(at)\ln(1+t)}{t}dt = \ln^2(at)\ln(1+t) +$$

$$+2\sum_{n=1}^\infty \frac{(-1)^n}{n}\int t^{n-1}(\ln(a)+\ln(t))\,dt = \ln^2(at)\ln(1+t) + 2Li_2(-t)\ln(at) -$$

$$-2Li_3(-t) + c = \ln^2(x)\ln\left(1+\frac{x}{a}\right) + 2Li_2\left(-\frac{x}{a}\right)\ln(x) - 2Li_3\left(-\frac{x}{a}\right) + c$$

$$\circledast J(a) = a\int \frac{\ln^2(at)}{1+t}dt = at\ln^2(at) - a\ln(1+t)\ln^2(at) - 2a\int \frac{\ln(at)(t-\ln(1+t))}{t}dt =$$

$$= at\ln^2(at) - a\ln(1+t)\ln^2(at) - 2a(\ln(at)-1)$$

$$-2a\sum_{n=1}^\infty \frac{(-1)^n}{n}\int t^{n-1}(\ln(a)+\ln(t))\,dt$$

$$= at\ln^2(at) - a\ln(1+t)\ln^2(at) + 2at - 2at\ln(at) + 2Li_3(-t) - 2aLi_2(-t)\ln^2(at) + c =$$

$$= 2aLi_3\left(-\frac{x}{a}\right) - 2aLi_2\left(-\frac{x}{a}\right)\ln(x) - a\ln^2(x)\ln\left(1+\frac{x}{a}\right) + 2x + x\ln^2(x)$$

$$-2x\ln(x) + c$$

$$\blacklozenge\Omega_1 = \frac{1}{2}\left(\Big|_0^1 I\left(\frac{1}{2}\right) - J\left(\frac{1}{2}\right)\right) = \frac{1}{2}(-2Li_3(-2) - Li_3(-2) - 2) = -\frac{3}{2}Li_3(-2) - 1$$

$$♦\Omega_2 = \int_0^1 \frac{(1-u)\ln^2(u)}{(1+u)}du = \int_0^1 \frac{\ln^2(u)}{1+u}du - \int_0^1 \frac{u\ln^2(u)}{1+u}du =$$

$$= \sum_{n=1}^{\infty} (-1)^n \int_0^1 u^2 \ln^2(u)du$$

$$- \int_0^1 u^{n+1}\ln^2(u)du = \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{(n+1)^3} - \frac{2}{(n+2)^3}\right) =$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^3} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = \frac{3}{2}\zeta(3) - 2 + \frac{3}{2}\zeta(3) = 3\zeta(3) - 2$$

$$\Omega = \int_0^{\infty} \frac{x\ln^2(1+x)}{(1+x)(2+x)(3+x)}dx = 2\Omega_1 - \Omega_2 = -3\mathrm{Li}_3(-2) - 3\zeta(3)$$

Note; $\zeta(3) - \text{Apery's constant}$

2331. Find:

$$\Omega = \int_0^1 \int_0^1 \frac{x^2 \ln(y)\ln(x)\ln(1+x^2)}{(1+y)^2} dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \int_0^1 \frac{x^2 \ln(y)\ln(x)\ln(1+x^2)}{(1+y)^2} dxdy$$

$$= \int_0^1 \frac{\ln(y)}{(1+y)^2}dy \times \int_0^1 x^2 \ln(x)\ln(1+x^2)\,dx = K \times M$$

$$K = \int_0^1 \frac{\ln(y)}{(1+y)^2}dy$$

$$\left\{\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x};\ \frac{\partial}{\partial x}\sum_{n=0}^{\infty}(-1)^n x^n = -\frac{1}{(1+x)^2}\ ;\ -\sum_{n=0}^{\infty}(-1)^n x^{n-1} = \frac{1}{(1+x)^2}\right\}$$

$$K = -\sum_{n=1}^{\infty}(-1)^n n \int_0^1 y^{n-1}\ln(y)\,dy = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = --\ln(2)$$

$$M = \int_0^1 x^2 \ln(x)\ln(1+x^2)\,dx = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{2n+2}\ln(x)\,dx =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n+3)^2} = \frac{1}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \frac{2}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+3} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+3)^2} =$$

$$=\frac{\ln(2)}{9}+\frac{2}{9}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+5}-\frac{16}{27}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}$$

$$\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+4}dx=\int_0^1\frac{x^4}{1+x^2}dx=\right\}$$

$$\left\{=\int_0^1(x^2-1)dx+\int_0^1\frac{1}{1+x^2}dx=\frac{1}{3}-1+\frac{\pi}{4}=\frac{\pi}{4}-\frac{2}{3}\right\}$$

$$M=-\frac{\ln(2)}{9}+\frac{\pi}{18}-\frac{4}{27}-\frac{16}{27}+\frac{G}{3}=-\frac{\ln(2)}{9}+\frac{\pi}{18}-\frac{20}{27}+\frac{2G}{3}$$

answer: $\Omega=K\times M=\frac{ln^2(2)}{9}-\frac{\pi\ln(2)}{18}+\frac{20\ln(2)}{27}-\frac{2\ln(2)}{3}G$

Note: G - Catalan's constant

2332. Prove that:

$$\int_1^{\infty}\frac{\ln^2(x)}{(1+x+x^2)^2}dx=\frac{16\sqrt{3}}{729}\pi^3-\frac{7}{81}\pi^2-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\int_1^{\infty}\frac{\ln^2(x)}{(1+x+x^2)^2}dx=\int_1^{\infty}\frac{(x-1)^2}{(x^3-1)^2}\ln^2(x)\,dx$$

$$\Rightarrow(x\to 1/x)\int_0^1\frac{(x-x^2)^2}{(1-x^3)^2}\ln^2(x)\,dx$$

$$\Rightarrow(x^3\to x)\ \frac{1}{27}\int_0^1\frac{\left(1-\sqrt[3]{x}\right)^2}{(1-x)^2}\ln^2(x)\,dx\Rightarrow$$

$$\frac{1}{27}\sum_{n\in\mathbb{N}}n\int_0^1\left(x^{n-1}+x^{n-\frac{1}{3}}-2x^{n-\frac{2}{3}}\right)\ln^2(x)\,dx\Rightarrow$$

$$\frac{2}{27}\sum_{n\in\mathbb{N}}n\left(\frac{1}{n^3}+\frac{27}{(3n+2)^3}-\frac{54}{(3n+1)^3}\right)=$$

$$=\frac{2}{27}\zeta(2)+\frac{2}{27}\zeta\left(2,\frac{2}{3}\right)-\frac{4}{81}\zeta\left(3,\frac{2}{3}\right)-\frac{4}{27}\zeta\left(2,\frac{1}{3}\right)+\frac{4}{81}\zeta\left(3,\frac{1}{3}\right)\Rightarrow$$

$$\frac{\pi^2}{81}-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)-\frac{2}{27}\left[\psi^{(1)}\left(\frac{1}{3}\right)+\psi^{(1)}\left(\frac{2}{3}\right)\right]+\frac{2}{81}\left[\psi^{(2)}\left(\frac{2}{3}\right)-\psi^{(2)}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow\frac{\pi^2}{81}-\frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)-\frac{2}{27}\frac{\pi^2}{\sin^2(\pi/3)}+\frac{4\pi^3}{81}\frac{\cot(\pi/3)}{\sin^2(\pi/3)}$$

$$\Rightarrow \int_1^\infty \frac{\ln^2(x)}{(1+x+x^2)^2}\,dx = \frac{16\sqrt{3}}{729}\pi^3 - \frac{7}{81}\pi^2 - \frac{2}{27}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{4}{27}\psi^{(1)}\left(\frac{2}{3}\right)$$

2333. Find:

$$\int_0^1\int_0^1 x^3\arctan^3(1-x^2)\ln(\ln^4(y))\,dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Bui Hong Suc-Vietnam

$$\therefore\ \gamma = -\int_0^\infty e^{-x}\ln x\,dx = -\int_0^1 \ln(-\ln x)\,dx$$

$$\therefore\ \int_0^{\frac{\pi}{4}} \ln\cos x\,dx = -\frac{\pi}{4}\ln 2 + \frac{G}{2}$$

$$\therefore\ \ln\cos x = -\ln 2 - \sum_{n=1}^\infty \frac{(-1)^n}{n}\cos 2nx$$

$$\therefore\ \int_0^{\frac{\pi}{4}} x\ln\cos x\,dx = \int_0^{\frac{\pi}{4}} x\left(-\ln 2 - \sum_{n=1}^\infty \frac{(-1)^n}{n}\cos 2nx\right)dx$$

$$= -\frac{x^2}{2}\ln 2\ \Big|_0^{\frac{\pi}{4}} - \sum_{n=1}^\infty \frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} x\cos 2nx\,dx$$

$$= -\frac{\pi^2\ln 2}{32} - \sum_{n=1}^\infty \frac{(-1)^n}{n}\ \frac{\pi n\sin\frac{n\pi}{2} + 2\cos\frac{n\pi}{2} - 2}{8n^2} = -\frac{\pi^2\ln 2}{32} + \frac{1}{4}\sum_{n=1}^\infty \frac{(-1)^n}{n^3} - \frac{\pi}{8}\sum_{n=1}^\infty \frac{(-1)^n\sin\frac{n\pi}{2}}{n^2} - \frac{1}{4}\sum_{n=1}^\infty \frac{(-1)^n\cos\frac{n\pi}{2}}{n^3} = -\frac{\pi^2\ln 2}{32} + \frac{1}{4}\cdot\frac{-3}{4}\zeta(3) - \frac{\pi}{8}\cdot(-G) - \frac{1}{4}\cdot\frac{1}{8}\cdot\frac{-3}{4}\zeta(3) = \frac{16\pi G - 21\zeta(3) - 4\pi^2\ln 2}{128}$$

$$\Omega = \int_0^1\int_0^1 x^3\arctan^3(1-x^2)\ln(\ln^4(y))\,dxdy =$$

$$\int_0^1 x^3\arctan^3(1-x^2)\,dx\int_0^1 \ln(-\ln(y))^4\,dy$$

$$= 4\int_0^1 x^2\,x\arctan^3(1-x^2)\,dx\int_0^1 \ln(-\ln(y))\,dy = 4A.(-\gamma) = -4\gamma A$$

$$\therefore\ A = \int_0^1 x^2\,x\arctan^3(1-x^2)\,dx = \frac{1}{2}\int_0^1 (1-v)\arctan^3 v\,dv$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{(1-\tan x)}{\cos^2 x}x^3\,dx = \frac{1}{2}\left(\int_0^{\frac{\pi}{4}} x^3\,d\tan x - \frac{1}{2}\int_0^{\frac{\pi}{4}} x^3\,d\tan^2 x\right)$$

$$*)J = \int_0^{\frac{\pi}{4}} x^3\,d\tan x = x^3\tan x\ \Big|_0^{\frac{\pi}{4}} + 3\int_0^{\frac{\pi}{4}} x^2\,d\ln\cos x =$$

$$= \frac{\pi^3}{64} + 3x^2\ln\cos x\ \Big|_0^{\frac{\pi}{4}} - 6\int_0^{\frac{\pi}{4}} x\ln\cos x\,dx$$

$$=\frac{\pi^3}{64}-\frac{3\pi^2}{32}\ln 2-6\int_0^{\frac{\pi}{4}} x\ln\cos x\,dx=\frac{\pi^3}{64}-\frac{3\pi^2}{32}\ln 2-6\,\frac{16\pi G-21\zeta(3)-4\pi^2\ln 2}{128}$$

$$=\frac{\pi^3+6\pi^2\ln 2+63\zeta(3)-48\pi G}{64}$$

$$*)K=\int_0^{\frac{\pi}{4}} x^3\,d\tan^2 x=x^3\tan^2 x\,\Big|_0^{\frac{\pi}{4}}-3\int_0^{\frac{\pi}{4}} x^2\tan^2 x\,dx=\frac{\pi^3}{64}-3\int_0^{\frac{\pi}{4}} x^2\left(\frac{1}{\cos^2 x}-1\right)dx$$

$$=\frac{\pi^3}{64}+3\int_0^{\frac{\pi}{4}} x^2\,dx-3\int_0^{\frac{\pi}{4}} x^2\,d\tan x=\frac{\pi^3}{64}+x^3\,\Big|_0^{\frac{\pi}{4}}-3\left(x^2\tan x\,\Big|_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}} x\tan x\,dx\right)$$

$$=\frac{\pi^3}{64}+\frac{\pi^3}{64}-3\left(\frac{\pi^2}{16}+2\int_0^{\frac{\pi}{4}} x d\ln\cos x\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}-6\left(x\ln\cos x\,\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\ln\cos x\,dx\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}+\frac{3\pi}{4}\ln 2+6\int_0^{\frac{\pi}{4}}\ln\cos x\,dx=\frac{\pi^3}{32}-\frac{3\pi^2}{16}+\frac{3\pi}{4}\ln 2+6\left(-\frac{\pi}{4}\ln 2+\frac{G}{2}\right)$$

$$=\frac{\pi^3}{32}-\frac{3\pi^2}{16}-\frac{3\pi}{4}\ln 2+3G$$

$\textit{Hence}:$

$$A=\frac{1}{2}\left(J-\frac{1}{2}K\right)=\frac{1}{2}\left(\frac{\pi^3+6\pi^2\ln 2+63\zeta(3)-48\pi G}{64}-\frac{1}{2}\left(\frac{\pi^3}{32}-\frac{3\pi^2}{16}-\frac{3\pi}{4}\ln 2+3G\right)\right)$$

$$=\frac{3}{2}\left(\frac{21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)}{64}\right)$$

$$\textit{Then}:\Omega=-4\gamma A=-4\gamma\frac{3}{2}\left(\frac{21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)}{64}\right)$$

$$=-\frac{3\gamma}{32}\big(21\zeta(3)-16G(\pi+2)+2\pi(\pi+\pi\ln 2+4\ln 2)\big)$$

Note : G "Catalan's" constant, ζ (3) "Apery's constant", γ "Euler-Mascheroni constant"

2334. Prove that:

$$\int_0^{\infty}\int_0^{\infty}\frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}\,dxdy=\frac{(\sqrt{2}-2)}{4}\sqrt{\pi}\csc\left(\frac{\pi}{8}\right)\Gamma\left(\frac{3}{4}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\int_0^\infty\int_0^\infty \frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}dxdy \Rightarrow \Re\int_0^\infty\int_0^\infty \frac{e^{i(x+y)}}{\sqrt{x\sqrt{y}}}dxdy$$

$$\left(\because \frac{\Gamma(z)}{x^z}=\int_0^\infty t^{z-1}e^{-xt}dt,\ \Re(z)>0\wedge\Re(x)>0\right)$$

$$\Rightarrow \Gamma(1/2)\Gamma(3/4)\Re\left(i^{5/4}\right)\Rightarrow \sqrt{\pi}\Gamma(3/4)\Re\left(e^{i5\pi/8}\right)$$

$$\Rightarrow \sqrt{\pi}\Gamma(3/4)\cos(5\pi/8)\Rightarrow -\sqrt{\pi}\Gamma(3/4)\sin^2(\pi/8)\csc(\pi/8)$$

$$\Rightarrow \int_0^\infty\int_0^\infty \frac{\cos(x+y)}{\sqrt{x\sqrt{y}}}dxdy=\frac{(\sqrt{2}-2)}{4}\sqrt{\pi}\csc\left(\frac{\pi}{8}\right)\Gamma\left(\frac{3}{4}\right)$$

2335. Prove that:

$$\int_0^1\int_0^1(\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy=$$

$$=\frac{\pi^2}{16}+\frac{\pi}{4}\ln(2)+2\ln\ln(2)+\gamma-\text{li}(2)-G$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\int_0^1\int_0^1(\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy$$

$$\underbrace{\int_0^1\int_0^1(\ln\ln(1+x)\,dxdy}_{:=\xi_1}+\underbrace{\int_0^1\int_0^1 arctan^2(1-y)dxdy}_{:=\xi_2}$$

$$\xi_1=\int_0^1\int_0^1(\ln\left(\ln(1+x)+arctan^2(1-y)\right)dxdy$$

$$\overset{I.B.P}{\Longrightarrow}\left(ln\ln(1+x)\int\frac{dx}{dy}xdx\right)_0^1-\int_0^1\frac{x}{(1+x)\ln(1+x)}dx\overset{1+x\to x}{\Longrightarrow} ln\ln(2)-\int_1^2\frac{dx}{x}+$$

$$+\underbrace{\int_1^2\frac{dx}{x\ln(x)}}_{\ln(x)\to x}\underset{(1)}{\overset{Note\ section}{\Longrightarrow}}\ln\ln(2)+\underbrace{Li(1)}_{li(1)-li(2)}+\ln\ln(2)-\lim_{x\to 0}\ln(x)$$

$$\underset{(2)}{\overset{Note\ section}{\Longrightarrow}}2ln\ln(2)+\gamma+\lim_{x\to 1}\ln\ln(x)-li(2)-\lim_{x\to 0}\ln(x)$$

$$\Rightarrow \xi_1 = \int_0^1 \int_0^1 \ln(1+x)\,dxdy - 2\ln\ln(2) + \gamma - li(2)$$

$$\xi_2 = \int_0^1 \int_0^1 arctan^2(1-y)dxdy \overset{1-y\to x}{\Longrightarrow} \int_0^1 \arctan^2(x)dx$$

$$\overset{I.B.P}{\Longrightarrow} \left(arctan^2(x)\int \frac{dx}{dy}dx\right)_0^1 - 2\int_0^1 \frac{x\tan^{-1}(x)}{1+x^2}dx$$

$$\overset{I.B.P}{\Longrightarrow} \frac{\pi^2}{16} - \left(\tan^{-1}(x)\int \frac{dx}{dy}\ln(1+x^2)\,dx\right)_0^1 + \underbrace{\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx}_{x\to\tan(x)}$$

$$\overset{tan^2(x)+1-sec^2(x)}{\Longrightarrow} \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln\cos(x)dx \underset{(3)}{\overset{Note\ section}{\Longrightarrow}} \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) +$$

$$+ 2\ln(2)\int_0^{\frac{\pi}{4}} dx + 2\sum_{n\in N}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} \cos(2nx)dx$$

$$\Rightarrow \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) + \sum_{n\in N}\frac{(-1)^n}{n}\left(\frac{\sin(2nx)}{n}\right)_0^{\pi/4} \overset{n-2n+1}{\Longrightarrow} \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) + \sum_{n\in N\cup\{0\}}\frac{(-1)^{n+1}}{(2n+1)^2}$$

$$\underset{(4)}{\overset{Note\ section}{\Longrightarrow}} \xi_2 = \int_0^1\int_0^1 arctan^2(1-y)dxdy = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) - G$$

Put the values of ξ_1 and ξ_2 in equation-(1)

$$\int_0^1\int_0^1 (\ln\left(\ln(1+x) + arctan^2(1-y)\right)dxdy = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) + 2\ln\ln(2) + \gamma - li(2) - G$$

Note: G – Catalans constant

γ – *Euler – Mascheroni constant*

2336. Prove that:

$$\sum_{x=2}^{\infty} \frac{(-1)^x \sin^2(x-1)\cos^2(x-1)}{(x-1)^2} = \frac{(\pi-2)^2}{8}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \sum_{n=2}^{\infty} \frac{(-1)^n \sin^2(n-1)\cos^2(n-1)}{(n-1)^2}$$

$$= -\frac{1}{4}\sum_{n=2}^{\infty}\frac{(-1)^n\sin^2(2n)}{n^2} = \frac{1}{16}\sum_{n=2}^{\infty}\frac{(-1)^n(e^{2in}-e^{-2in})^2}{n^2} =$$

$$= \frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^n(e^{4in}-2+e^{-4in})}{n^2} = \frac{1}{16}\sum_{n=2}^{\infty}\frac{(-1)^n(e^{4in}+e^{-4in})}{n^2} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} =$$

We know $\to Li_2(z) + Li_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{\log(-z)}{2} = \frac{1}{16}\left(Li_2(-e^{4i}) + Li_2\left(-\frac{1}{e^{4i}}\right)\right) + \frac{\pi^2}{96} =$

$$=\frac{1}{16}\left(-\frac{\pi^2}{6}-\frac{\log^2(e^{4i})}{2}\right)+\frac{\pi^2}{96}=-\frac{\pi^2}{96}+\frac{\pi^2}{96}-\frac{\log^2(\cos(4)-i\sin(4))}{32}$$

$$=\frac{\log^2(\cos(2\pi-4)-i\sin(2\pi-4))}{32}=$$

$$=\frac{\log^2(e^{2\pi-4i})}{32}=\frac{(2\pi-4i)^2}{32}=\frac{(\pi-2)^2}{8}$$

2337. If $f(x)=\sum_{k=0}^{x}\frac{k}{2k+e^2}$ Prove that

$$\int_0^1 f(1-x)dx=\frac{1}{4}-\frac{e^2}{4}\left(\log\left(\frac{e^2}{2}+1\right)-\psi^{(0)}\left(\frac{e^2}{2}+1\right)\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$f(x)=\sum_{k=0}^{x}\frac{k}{2k+e^2}=\frac{1}{2}\sum_{k=0}^{x}\frac{k+\frac{e^2}{2}-\frac{e^2}{2}}{k+\frac{e^2}{2}}=\frac{1}{2}\left(\sum_{k=0}^{x}\left(1-\frac{-\frac{e^2}{2}}{k+\frac{e^2}{2}}\right)\right)=\frac{1}{2}\left(x+1-\frac{e^2}{2}\sum_{k=0}^{x}\frac{1}{k+\frac{e^2}{2}}\right)=$$

$$=\frac{1}{2}(x+1-\frac{e^2}{2}\left(\psi\left(x+\frac{e^2}{2}+1\right)-\psi\left(\frac{e^2}{2}\right)\right)$$

$$=\frac{1}{2}(x+1-\frac{e^2}{2}\left(\psi\left(x+\frac{e^2}{2}+1\right)-\psi\left(\frac{e^2}{2}+1\right)++\frac{e^2}{2}\right)$$

$$=\frac{1}{2}(x-\frac{e^2}{2}\psi\left(x+\frac{e^2}{2}+1\right)+\psi\left(\frac{e^2}{2}+1\right)+\frac{e^2}{2}$$

$$\Omega=\int_0^1 f(1-x)dx=|1-x=x|=\int_0^1 f(x)dx=\frac{1}{2}\int_0^1 xdx-\frac{e^2}{4}\int_0^1\psi\left(x+\frac{e^2}{2}+1\right)dx+$$

$$+\frac{e^2}{2}\psi\left(\frac{e^2}{2}+1\right)\int_0^1 dx=\frac{1}{4}-\frac{e^2}{4}\left[\log\Gamma(x+\frac{e^2}{2}+1)\right]_0^1+\frac{e^2}{2}\psi\left(\frac{e^2}{2}+1\right)$$

$$=\frac{1}{4}-\log\left(\frac{\Gamma\left(2+\frac{e^2}{2}\right)}{\Gamma\left(1+\frac{e^2}{2}\right)}\right)+\frac{e^2\psi\left(\frac{e^2}{2}+1\right)}{4}$$

Notes: $\psi(x+n+1)-\psi(x)=\sum_{k=0}^{x}\frac{k}{k+x}$; $\psi(x+1)-\psi(x)=\frac{1}{x}$

answer: $\int_0^1 f(1-x)dx=\frac{1}{4}-\frac{e^2}{4}\left(\log\left(\frac{e^2}{2}+1\right)-\frac{e^2}{4}\psi\left(\frac{e^2}{2}+1\right)\right)$

2338. **Prove the below closed form:**

$$I=\int_0^1\int_0^1\left(\frac{x+y}{x-y}\right)\tan^{-1}\left(\frac{x-y}{x+y}\right)dx\,dy=2G-\frac{\pi}{4}\log(2)-\frac{\log(2)}{2}$$

Where, G is a Catalan's constant.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\left(\frac{x+y}{x-y}\right)\tan^{-1}\left(\frac{x-y}{x+y}\right)dx\,dy=\int_0^1\int_0^1\left(\frac{1+y/x}{1-y/x}\right)\tan^{-1}\left(\frac{1-y/x}{1+y/x}\right)dx\,dy$$

Let $y/x\to m$

$$I=\int_0^1 x\int_0^{\frac{1}{x}}\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\,dx\overset{IBP}{=}$$

$$=\left[\frac{x^2}{2}\int_0^{\frac{1}{x}}\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\right]_0^1+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\frac{1}{2}\int_0^1\left(\frac{1+m}{1-m}\right)\tan^{-1}\left(\frac{1-m}{1+m}\right)dm\Bigg]_{m\to x}+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\frac{1}{2}\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}\left(\frac{1-x}{1+x}\right)dx+\frac{1}{2}\int_0^1\left(\frac{1+x}{x-1}\right)\tan^{-1}\left(\frac{x-1}{1+x}\right)dx=$$

$$=\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}\left(\frac{1-x}{1+x}\right)dx=\frac{\pi}{4}\int_0^1\left(\frac{1+x}{1-x}\right)dx-\int_0^1\left(\frac{1+x}{1-x}\right)\tan^{-1}(x)\,dx=$$

$$=\frac{\pi}{4}\int_0^1\left(\frac{1+x}{1-x}\right)dx-2\int_0^1\frac{\tan^{-1}(x)}{1-x}dx+\int_0^1\tan^{-1}(x)\,dx$$

Use the integration by parts formula

$$I=\frac{\pi}{4}\{-2\log(1-x)]_0^1-1\}-2\left\{-\log(1-x)\tan^{-1}(x)]_0^1+\int_0^1\frac{\log(1-x)}{1+x^2}dx\right\}+\int_0^1\tan^{-1}(x)\,dx$$

$$=\log(1-x)\left\{-\frac{\pi}{2}+2\tan^{-1}(x)\right\}\Big]_0^1-\frac{\pi}{4}-2\int_0^1\frac{\log(1-x)}{1+x^2}dx+\int_0^1\tan^{-1}(x)\,dx$$

$$=-\frac{\pi}{4}-2\int_0^1\frac{\log(1-x)}{1+x^2}dx+\int_0^1\tan^{-1}(x)\,dx$$

$$=-\frac{\pi}{4}-2\left\{\frac{\pi}{8}\log(2)-G\right\}+\left\{-\frac{\log(2)}{2}+\frac{\pi}{4}\right\}=2G-\frac{\pi}{4}\log(2)-\frac{\log(2)}{2}$$

2339. **If $0<a\le b$ then:**

$$\int_a^b e^{x^2}\,dx\ge(b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

Proposed by Daniel Sitaru – Romania

Solution by George Florin Șerban – Romania

$$e^x\ge x+1,(\forall)x>0\Rightarrow e^{x^2}\ge x^2+1$$

$$\Rightarrow\int_a^b e^{x^2}\,dx\ge\int_a^b(x^2+1)\,dx=\frac{b^3-a^3}{3}+b-a=$$

$$=\frac{(b-a)(b^2+ab+a^2)}{3}+b-a=(b-a)\cdot\left(\frac{a^2+ab+b^2}{3}+1\right)\ge$$

$$\ge(b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

$$b-a\ge0,\frac{a^2+ab+b^2}{3}+1\ge\sqrt[3]{a^2+ab+b^2},\qquad S=a^2+ab+b^2>0$$

$$\frac{S+3}{3}\ge\sqrt[3]{S}\Rightarrow(S+3)^3\ge27S\Rightarrow S^3+9S^2+27S-27S\ge0$$

$\Rightarrow S^3+9S^2>0$, true, $(\forall)S>0$. Then: $\int_a^b e^{x^2}\,dx\ge(b-a)\sqrt[3]{a^2+ab+b^2}$

Equality holds for $a=b$.

2340. **Prove that**

$$I=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(xy)}{(1+x)^2(1+y)}dx\,dy=\frac{\log(2)}{96}(48G-12\pi\log(2)-5\pi^2)$$

Where, G is Catalan's constant

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(xy)}{(1+x)^2(1+y)}dx\,dy=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2(1+y)}dx\,dy+\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(y)}{(1+x)^2(1+y)}dx\,dy$$

$$=I_1+I_2$$

$$I_1=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2(1+y)}dx\,dy$$

$$=\log(2)\int_0^1\frac{\tan^{-1}(x)\log(x)}{(1+x)^2}dx\overset{IBP}{=}\log(2)\left\{-\frac{\pi}{4}\log(2)+\frac{1}{2}\int_0^1\frac{\log(x)}{1+x}dx\right.$$

$$\left.+\int_0^1\frac{\log(1+x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\log(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{x\log(x)}{1+x^2}dx\right\}$$

$$=\log(2)\left\{\frac{1}{2}G-\frac{\pi}{8}\log(2)-\frac{3}{16}\zeta(2)\right\}$$

$$I_2=\int_0^1\int_0^1\frac{\tan^{-1}(x)\log(y)}{(1+x)^2(1+y)}dx\,dy=-\frac{\zeta(2)}{2}\int_0^1\frac{\tan^{-1}(x)}{(1+x)^2}dx=-\frac{1}{8}\log(2)\,\zeta(2)$$

$$I=I_1+I_2=\frac{\log(2)}{96}\left(48G-12\pi\log(2)-5\pi^2\right)$$

2341. Find:

$$\Omega=\int_0^2\int_0^2 max\left(min\left(x,\frac{1}{y},\frac{xy+1}{x}\right)\right)dxdy$$

Proposed by Daniel Sitaru-Romania

Solution by Ahmed Salem-Tunisia

Case 1:

$$x\le\frac{1}{y}\Rightarrow xy\le 1$$

$$min\left(x,\frac{1}{y},\frac{xy+1}{x}\right)=min\left(x,\frac{xy+1}{x}\right)\le\sqrt{x\cdot\frac{xy+1}{x}}=\sqrt{xy+1}\le\sqrt{1+1}=\sqrt{2}$$

Case 2:

$$x\ge\frac{1}{y}\Rightarrow xy\ge 1\Rightarrow\frac{1}{xy}\le 1$$

$$min\left(x,\frac{1}{y},\frac{xy+1}{x}\right)=min\left(\frac{1}{y},\frac{xy+1}{x}\right)\le\sqrt{\frac{1}{y}\cdot\frac{xy+1}{x}}=\sqrt{\frac{xy+1}{xy}}=$$

$$= \sqrt{1 + \frac{1}{xy}} \leq \sqrt{1 + 1} = \sqrt{2}$$

$$max\left(min\left(x, \frac{1}{y}, \frac{xy + 1}{x}\right)\right) = \sqrt{2}$$

$$\Omega = \int_0^2 \int_0^2 max\left(min\left(x, \frac{1}{y}, \frac{xy + 1}{x}\right)\right) dxdy = \int_0^2 \int_0^2 \sqrt{2}\, dxdy = 4\sqrt{2}$$

2342. **If $a, b \in \mathbb{R}, a \leq b, f: [a, b] \to (0, \infty), f -$ continuous then:**

$$3\int_a^b f(x)\, dx + \left(\int_a^b \frac{1}{f(x)} dx\right)^3 \geq 4(b - a)^{\frac{3}{2}}$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

The function $\ln$ is concave so:

$$\ln\left(\frac{3}{4}\int_a^b f(x)\, dx + \frac{1}{4}\left(\int_a^b \frac{1}{f(x)} dx\right)^3\right) \geq \frac{3}{4}\ln\left(\int_a^b f(x)\, dx\right) + \frac{1}{4}\ln\left(\left(\int_a^b \frac{1}{f(x)} dx\right)^3\right)$$

i.e.:

$$\ln\left(\frac{3}{4}\int_a^b f(x)\, dx + \frac{1}{4}\left(\int_a^b \frac{1}{f(x)} dx\right)^3\right) \geq \frac{3}{4}\ln\left(\left(\int_a^b f(x)\, dx\right)\left(\int_a^b \frac{1}{f(x)} dx\right)\right)$$

So (since exp is growing):

$$\frac{3}{4}\int_a^b f(x)\, dx + \frac{1}{4}\left(\int_a^b \frac{1}{f(x)} dx\right)^3 \geq \left(\left(\int_a^b f(x)\, dx\right)\left(\int_a^b \frac{1}{f(x)} dx\right)\right)^{\frac{3}{4}}$$

The Cauchy – Schwarz inequality gives:

$$\left(\int_a^b \left(\sqrt{f(x)}\right)^2 dx\right)\left(\int_a^b \left(\frac{1}{\sqrt{f(x)}}\right)^2 dx\right) \geq \left(\int_a^b \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} dx\right)^2$$

i.e.:

$$\left(\int_a^b f(x)\, dx\right)\left(\int_a^b \frac{1}{f(x)} dx\right) \geq (b - a)^2$$

So:

$$\frac{3}{4}\int_a^b f(x)\,dx+\frac{1}{4}\left(\int_a^b \frac{1}{f(x)}dx\right)^3 \geq (b-a)^{\frac{3}{2}}$$

Finally:

$$3\int_a^b f(x)\,dx+\left(\int_a^b \frac{1}{f(x)}dx\right)^3 \geq 4(b-a)^{\frac{3}{2}}$$

Equality holds for $a=b$.

2343. **If $0 \leq x \leq 1, n \geq 1$ then:**

$$\sum_{k=1}^{n}\int_0^1 (cosx)^{k-1}\cdot(sinx)^k\,dx < 2\left(1-\frac{1}{2^n}\right)$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Daniel Sitaru-Romania

$$x\in[0,1]\subset\left[0,\frac{\pi}{2}\right)\Rightarrow sinx\geq 0, cosx>0$$

$$(cosx)^{k-1}\cdot(sinx)^k = sinx\cdot(cosx)^{k-1}\cdot(sinx)^{k-1}=$$

$$= sinx\cdot(sinx\cdot cosx)^{k-1}= sinx\cdot(sin^2x\cdot cos^2x)^{\frac{k-1}{2}}\leq$$

$$\leq sinx\cdot\left(\left(\frac{sin^2x+cos^2x}{2}\right)^2\right)^{\frac{k-1}{2}} = sinx\cdot\left(\left(\frac{1}{2}\right)^2\right)^{\frac{k-1}{2}} = sinx\cdot\left(\frac{1}{2}\right)^{k-1}<\left(\frac{1}{2}\right)^{k-1}$$

$$\sum_{k=1}^{n}\int_0^1 (cosx)^{k-1}\cdot(sinx)^k\,dx < \sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k-1} = \frac{\left(\frac{1}{2}\right)^n-1}{\frac{1}{2}-1}=2\left(1-\frac{1}{2^n}\right)$$

2344. **Prove that:**

$$I=\int_0^\infty\int_0^\infty \frac{e^{-x}x^4\log(x)\,(\log(y)-\log(x))}{(x^2+y^2)^2}dxdy=\frac{\pi}{4}(\gamma-1)$$

Where, γ is Euler-Mascheroni constant

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

Let y/x=t

$$I=\int_0^{\infty}\int_0^{\infty}\frac{e^{-x}x^4\log(x)\left(\log(y)-\log(x)\right)}{(x^2+y^2)^2}dxdy=\int_0^{\infty}x\log(x)e^{-x}\,dx\int_0^{\infty}\frac{\log(t)}{(1+t^2)^2}dt$$

$$=I_1*I_2$$

$$I_1=\int_0^{\infty}x\log(x)e^{-x}\,dx\overset{IBP}{=}-xe^{-x}\log(x)\Big|_0^{\infty}+\int_0^{\infty}e^{-x}\log(x)\,dx+\int_0^{\infty}e^{-x}\,dx$$

$$=\int_0^{\infty}e^{-x}\log(x)\,dx+1=-\gamma+1$$

$$I_2=\int_0^{\infty}\frac{\log(t)}{(1+t^2)^2}dt=\int_0^{1}\frac{\log(t)}{(1+t^2)^2}dt+\int_1^{\infty}\frac{\log(t)}{(1+t^2)^2}dt\Bigg]_{t\to 1/t}$$

$$=\int_0^{1}\frac{\log(t)}{(1+t^2)^2}dt-\int_0^{1}\frac{t^2\log(t)}{(1+t^2)^2}dt$$

$$=\int_0^{1}\frac{(1-t^2)\log(t)}{(1+t^2)^2}dt\overset{IBP}{=}\frac{t}{1+t^2}\log(t)\Big|_0^1-\int_0^1\frac{dt}{1+t^2}=-\int_0^1\frac{dt}{1+t^2}=-\frac{\pi}{4}$$

$$I=I_1*I_2=\frac{\pi}{4}(\gamma-1)$$

2345. Find:

$$X=\int_0^1\frac{ln^2(x+1)}{(x+1)(x+3)}dx\quad Y=\int_0^1\frac{x ln^2(x+1)}{(x+1)(x+3)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$X+Y=\int_0^1\frac{ln^2(x+1)}{x+3}dx\overset{x=2t-1}{\cong}\int_{\frac{1}{2}}^1\frac{ln^2(2t)}{t+1}dx=$$

$$=\left([\ln(1+t)\,ln^2(2t)+2\ln(2t)\,Li_2(-t)-2Li_3(-t)]\right)_{\frac{1}{2}}^{1}$$

$$X+Y=ln^3(2)+2ln(2)Li_2(-1)-2Li_3(-1)+2Li_3\left(-\frac{1}{2}\right)=ln^3(2)-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2Li_3\left(-\frac{1}{2}\right)$$

$$X-Y=\int_0^1\frac{(1-x)ln^2(x+1)}{(x+1)(x+3)}dx=\int_0^1\frac{ln^2(x+1)}{x+1}dx-2\int_0^1\frac{ln^2(x+1)}{x+3}dx$$

$$X-Y=\frac{1}{3}ln^3(2)-2ln^3(2)+\frac{\pi^2}{3}\ln(2)-3\zeta(3)-4Li_3(-\frac{1}{2})$$

$$X-Y=-\frac{5}{3}ln^3(2)+\frac{\pi^2}{3}\ln(2)-3\zeta(3)-4Li_3(-\frac{1}{2})$$

$$X+Y=ln^3(2)-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2Li_3(-\frac{1}{2})$$

$$X=-\frac{1}{3}ln^3(2)+\frac{\pi^2}{12}\ln(2)-\frac{3}{4}\zeta(3)-Li_3(-\frac{1}{2})$$

$$Y=\frac{4}{3}ln^3(2)-\frac{\pi^2}{4}\ln(2)+\frac{9}{4}\zeta(3)+3Li_3(-\frac{1}{2})$$

We have also : $Li_3\left(-\frac{1}{2}\right)=\frac{1}{4}Li_3(-\frac{1}{4})-\frac{7}{8}\zeta(3)-\frac{1}{6}ln^3(2)+\frac{\pi^2}{12}\ln(2)$

$$\int_0^1\frac{ln^2(x+1)}{(x+1)(x+3)}dx=\frac{1}{24}(-Li_3(\frac{1}{4})+3\zeta(3)-4ln^3(2))$$

$$\int_0^1\frac{xln^2(x+1)}{(x+1)(x+3)}dx=3Li_3\left(-\frac{1}{2}\right)+\frac{9}{4}\zeta(3)+\frac{4}{3}ln^3(2)-\frac{\pi^2}{4}\ln(2)$$

2346. Find:

$$\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx=\frac{1}{2}\int_0^1\frac{Li_2(x)\ln(1+x)}{x(1+x)}dx=\frac{1}{2}\underbrace{\int_0^1\frac{Li_2(x)\ln(1+x)}{x}dx}_{A}-\frac{1}{2}\underbrace{\int_0^1\frac{Li_2(x)\ln(1+x)}{1+x}}_{B}$$

$A=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}Li_2(x)dx=\int_0^1 x^{n-1}Li_2(x)=(\left[\frac{1}{n}x^nLi_2(x)\right])_0^1+\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$ $\quad A=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{5}{4}\zeta(4)+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=2Li_4\left(\frac{1}{2}\right)-\frac{11}{4}\zeta(4)+\frac{7}{4}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}ln^4(2)$$

$A=2Li_4\left(\frac{1}{2}\right)-\frac{3}{2}\zeta(4)+\frac{7}{4}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}ln^4(2)$, $B=\left[\frac{1}{2}Li_2(x)ln^2(1+x)\right]_0^1+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx=\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx$ $\quad ab^2=\frac{1}{6}(a+b)^3+\frac{1}{6}(a-b)^3-\frac{a^2}{3}$

$$\text{Ln}(1-x)ln^2(1+x)=\frac{1}{6}ln^3(1-x^2)+\frac{1}{6}ln^3\left(\frac{1-x}{1+x}\right)-\frac{1}{3}ln^3(1-x)$$

$$B=\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}\underbrace{\int_0^1\frac{ln^3(1-x^2)}{x}dx}_{1-x^2\to x}+\frac{1}{12}\underbrace{\int_0^1\frac{\left(\frac{1-x}{1+x}\right)}{x}dx}_{\frac{1-x}{1+x}\to x}-\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(1-x)}{x}dx}_{1-x\to x}=\frac{1}{2}\zeta(2)ln^2(2)+$$

$$\frac{1}{6}\sum_{n=1}^{\infty}\int_0^1 x^{2n-2}ln^3(x)dx-\frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}ln^3(x)dx=\frac{1}{2}\zeta(2)ln^2(2)-\frac{3}{16}\zeta(4)$$

$$\Omega=Li_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)-\frac{21}{32}\zeta(4)+\frac{1}{24}ln^4(2)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^{n-1}Li_2(x)dx=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\left[\frac{H_n}{n^2}-\frac{\zeta(2)}{n}\right]=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{n^2}=-\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n-1}+\frac{1}{n}\right)}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx=$$

$$-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}x^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^nx^n}{n^2}\right)dx=-\frac{1}{2}\int_0^1\frac{\ln(1-x)}{x}ln^2(1+x^2)dx+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=J-K$$

$$J=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^{n-1}ln^2(x)dx=-$$

$$\frac{1}{2}\int_0^1\frac{\ln(1+x)ln^2(x)}{x(1+x)}dx=\frac{1}{2}\Big(\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-x)ln^2(x)}{x}dx+\int_0^{\frac{1}{2}}\frac{ln^3(1-x)}{x}dx}_{J_1}-\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(x)ln^2(1-x)}{x}dx}_{J_2}=0.5J_1-J_2$$

$$J_1=\int_0^{\frac{1}{2}}\frac{ln^2(x)\ln(1-x)}{x}dx+\int_0^{\frac{1}{2}}\frac{ln^3(1-x)}{x}dx=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{1}{2}}x^{n-1}ln^2(x)dx+\int_{\frac{1}{2}}^1\frac{ln^3(x)}{1-x}dx$$

$$=\frac{1}{3}ln^4(2)+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(x)}{1-x}dx+\int_0^1\frac{ln^3(x)}{1-x}dx+\int_{\frac{1}{2}}^0\frac{ln^3(x)}{1-x}dx=4Li_4\left(\frac{1}{2}\right)-6\zeta(4)+\frac{7}{2}\zeta(3)\ln(2)-\zeta(2)ln^2(2)+\frac{2}{3}ln^4(2)$$

$$J_2=\int_0^{\frac{1}{2}}\frac{\ln(x)ln^2(1-x)}{x}dx=\frac{1}{4}ln^4(2)+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{1-x}dx$$
$$=\frac{1}{4}ln^4(2)$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1 x^n ln^2(x)dx=\frac{1}{4}ln^4(2)+2\zeta(4)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}$$
$$=\frac{1}{4}ln^4(2)+2\zeta(4)-\frac{9}{4}\zeta(4)=\frac{1}{4}ln^4(2)-\frac{\pi^4}{360}$$
$$J=\frac{1}{2}J_1-J_2=2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln(2)\zeta(3)+\frac{ln^4(2)}{12}-\frac{\pi^2}{12}ln^2(2)-\frac{11\pi^4}{360}$$
$$K=-\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx}_{ab^2=>\frac{1}{6}(a+b)^3+\frac{1}{6}(a-b)^3-\frac{a^3}{3}}=\frac{1}{8}\int_0^1\frac{ln^3(x)}{1-x}dx-\frac{1}{12}\underbrace{\int_0^1\left(ln^3\left(\frac{1-x}{1+x}\right)\right)/xdx}_{\frac{1-x}{1+x}=>x}=$$
$$\frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 x^n ln^3(x)dx-\frac{1}{6}\int_0^1\frac{ln^3(x)}{1-x^2}dx=-\frac{\pi^4}{480}$$
$$\Omega_1=J-K=2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\zeta(3)\ln(2)+\frac{1}{12}ln^4(2)-\frac{\pi^2}{12}ln^2(2)-\frac{41\pi^4}{1440}$$
$$\Omega_2=\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}=\zeta(2)\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^{n-1}dx=$$
$$-\zeta(2)\underbrace{\int_0^1\frac{\ln(1+x)}{x(1+x)}dx}_{\left\{\frac{1}{x(1+x)}=\frac{1}{x}-\frac{1}{1+x}\right\}}=\frac{1}{2}\zeta(2)ln^2(2)-\frac{\zeta^2(2)}{2}=\frac{\pi^2}{12}ln^2(2)-\frac{\pi^4}{72}$$

Answer: $\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx=\frac{1}{2}(\Omega_1-\Omega_2)=Li_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\ln(2)+\frac{1}{24}ln^4(2)-\frac{\pi^2}{12}ln^2(2)-\frac{7\pi^4}{960}$

Note : ζ(3) --- Apery's constant

2347. Find:

$$\Omega=\int_0^1\frac{\sqrt{x}\ln(\ln(x))}{1+x}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\sqrt{x}\ln(\ln(x))}{1+x}dx\overset{x\to x^2}{=}2\int_0^1\frac{x^2\ln(2\ln(x))}{1+x^2}dx=2\ln(-2)\int_0^1\frac{x^2}{1+x^2}dx+2\int_0^1\frac{x^2\ln(-\ln(x))}{1+x^2}dx$$

$$\int_0^1\frac{x^2}{1+x^2}dx=\int_0^1\left(1-\frac{1}{1+x^2}\right)dx=1-\frac{\pi}{4}\ ,\ \ln(-2)=2\ln(i)+\ln(2)=i\pi+\ln(2)$$

$$\Omega=\ln(4)-\frac{\pi}{2}\ln(2)-\frac{1}{2}i\pi(\pi-4)+2\int_0^1\ln(-\ln(x))\,dx-2\int_0^1\frac{\ln(-\ln(X))}{1+x^2}dx$$

$$\int_0^1 \ln(-\ln(x))\,dx \overset{x\to e^{-t}}{\cong} \int_0^\infty \ln(t)\,e^{-t}dt = -\gamma$$

Malmsten's integral: $M(\varphi)=\int_0^1 \frac{\ln(-\ln(x))}{1+2x\cos(\varphi)}dx = \frac{\pi}{2\sin(\varphi)} ln\left\{\frac{(2\pi)^{\frac{\varphi}{\pi}}\Gamma(\frac{1}{2}+\frac{\varphi}{2\pi})}{\Gamma(\frac{1}{2}-\frac{\varphi}{2\pi})}\right\}$ $-\pi<\varphi<\pi$

$$M(\tfrac{\pi}{2})=\int_0^1 \frac{\ln(-\ln(X))}{1+x^2}dx = \frac{\pi}{2} ln\left\{\frac{\sqrt{2\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right\} = \frac{\pi}{4}\ln(2)+\frac{\pi}{4}\ln(\pi)-\pi ln\Gamma\left(\frac{1}{4}\right)+\frac{\pi}{2} ln\underbrace{\left\{\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4})\right\}}_{\pi\sqrt{2}}$$

$$\int_0^1 \frac{\ln(-\ln(x))}{1+x^2}dx = \frac{\pi}{2}\ln(2)+\frac{3\pi}{4}\ln(\pi)-\pi ln\Gamma(\frac{1}{4})$$

$$\Omega=\ln(4)-\frac{\pi}{2}\ln(2)-\frac{1}{2}i\pi(\pi-4)-2\gamma-\pi\ln(2)-\frac{3\pi}{2}\ln(\pi)+2\pi\ln(\Gamma\left(\frac{1}{4}\right))$$

$$\int_0^1 \frac{\sqrt{x}\ln(\ln(x))}{1+x}dx = \ln(4)+2\pi ln\Gamma\left(\frac{1}{4}\right)-\frac{3\pi}{2}\ln(2\pi)-\frac{1}{2}i\pi(\pi-4)-2\gamma$$

2348. Prove that:

$$\Omega = \int_0^1 \frac{log^2(x)}{(1+x+x^2)(1+x+x^2+x^3)}dx = \frac{21}{32}\zeta(3)+\frac{8\pi^3}{81\sqrt{3}}-\frac{\pi^3}{32}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 \frac{log^2(x)}{(1+x+x^2)(1+x+x^2+x^3)}dx = \int_0^1 \frac{log^2(x)}{1+x+x^2}dx - \int_0^1 \frac{xlog^2(x)}{1+x+x^2+x^3}dx =$$

$$\underbrace{\int_0^1 \frac{(1-x)log^2(x)}{1-x^3}dx}_{A}-\underbrace{\int_0^1 \frac{x(1-x)log^2(x)}{1-x^4}dx}_{B}$$

$$A \overset{x^3=>x}{\cong} \frac{1}{27}\int_0^1 \frac{(x^{\frac{1}{3}-1}-x^{\frac{2}{3}-1})log^2(x)}{1-x}dx = \frac{1}{27}(\psi^{(2)}\left(\frac{2}{3}\right)-\psi^{(2)}\left(\frac{1}{3}\right))$$

$$A=\frac{\pi}{27}\lim_{Z\to\frac{1}{3}}\frac{d^2}{z^2d}\cot(\pi z) = \frac{2\pi^3}{27}\lim_{z\to\frac{1}{3}}\frac{\cot(\pi z)}{sin^2(\pi z)} = \frac{8\pi^3}{81\sqrt{3}}$$

$$B \overset{x^4\to x}{\cong} \frac{1}{64}\int_0^1 \frac{(x^{\frac{1}{2}-1}-x^{\frac{3}{4}-1})log^2(x)}{1-x}dx = \frac{1}{64}(\psi^{(2)}\left(\frac{3}{4}\right)-\psi^{(2)}\left(\frac{1}{2}\right))$$

$$B=\frac{1}{64}\left(2\pi^3-42\zeta(3)\right) = \frac{\pi^3}{32}-\frac{21}{32}\zeta(3)$$

2349. Find:

$$\int_{-1}^{1}\frac{xln(x)}{x^4+x^2+1}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_{-1}^{1}\frac{xlog(x)}{x^4+x^2+1}dx=\int_{0}^{1}\frac{xlog(x)}{x^4+x^2+1}dx+\int_{-1}^{0}\frac{xlog(x)}{x^4+x^2+1}dx=\int_{0}^{1}\frac{xlog(x)}{x^4+x^2+1}dx+\int_{0}^{1}\frac{i\pi x+xln(x)}{x^4+x^2+1}dx=$$

$$-\pi i\int_0^1\frac{x}{x^4+x^2+1}dx=-i\pi\int_0^1\frac{x-x^3}{1-x^6}dx=-i\pi\sum_{n=0}^{\infty}\left(\int_0^1 x^{6n+1}dx-\int_0^1 x^{6n+3}dx\right)=$$

$$-\pi i\sum_{n=0}^{\infty}\left(\frac{1}{6n+2}-\frac{1}{6n+4}\right)=-\frac{i\pi}{2}\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+2)}=-\frac{i\pi}{18}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{3})(n+\frac{2}{3})}=$$

$$-\frac{i\pi}{18}\frac{\psi_0\left(\frac{2}{3}\right)-\psi_0\left(\frac{1}{3}\right)}{\frac{1}{3}}=-\frac{i\pi}{6}\pi\cot\left(\frac{\pi}{3}\right)=-\frac{i\pi^2}{6\sqrt{3}}$$

We know - $\psi_0(1-x)-\psi_0(x)=\pi\cot(\pi x)$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\int_{-1}^{1}\frac{xln(x)}{x^4+x^2+1}dx=\overbrace{\int_{-1}^{0}\frac{xln(x)}{x^4+x^2+1}dx}^{x\to-x}+\int_{0}^{1}\frac{xln(x)}{x^4+x^2+1}dx=\int_{1}^{0}\frac{xln(-x)}{x^4+x^2+1}dx+\int_{0}^{1}\frac{xln(x)}{x^4+x^2+1}dx=$$

$$-\int_0^1\frac{x\{ln(-1)+\ln(x)\}}{x^4+x^2+1}dx+\int_0^1\frac{xln(x)}{x^4+x^2+1}dx=-\int_0^1\frac{xln(e^{i\pi})}{x^4+x^2+1}dx-\int_0^1\frac{xln(x)}{x^4+x^2+1}dx+\int_0^1\frac{xln(x)}{x^4+x^2+1}dx=$$

$$-i\pi\int_0^1\frac{x}{(x^2+\frac{1}{2})^2+\frac{3}{4}}dx=-\frac{i\pi}{\sqrt{3}}\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)\Big|_0^1=-\frac{i\pi}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=-\frac{i\pi^2}{6\sqrt{3}}$$

2350. Find:

$$\int_1^{\infty}\frac{\ln(1+x)\ln(1+x^2)}{(1+x)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pham Duc Nam-Vietnam

$$I=\int_1^{\infty}\frac{\ln(1+x)\ln(x^2+1)}{(1+x)^2}dx$$

$$=\frac{1}{2}\int_1^{\infty}\ln(1+x)\ln(1+x^2)\,d\left(\frac{x-1}{x+1}\right),t=\frac{x-1}{x+1}=>x=\frac{1+t}{1-t}$$

$$I=\frac{1}{2}\int_0^1(\ln(2)-\ln(1-t))\left(\ln(2)+\ln(t^2+1)-2\ln(1-t)\right)dt=$$

$$\frac{1}{2}\int_0^1\left(ln^2(2)+\ln(2)\ln(t^2+1)-3\ln(2)\ln(1-t)-\ln(1-t)\ln(1+t^2)+2ln^2(1-t)\right)dt$$

By integration by parts, we easily find:

$$\int \ln(1+x^2)\,dx = x(\ln(1+x^2)-2)+\tan^{-1}(x)+C$$

$$\int \ln^2(1-x)dx = 2x+(x-1)\ln(1-x)(\ln(1-x)-2)+C$$

$$I=\frac{1}{2}\ln^2(2)+\frac{1}{2}\ln^2(2)-\ln(2)+\frac{1}{4}\pi\ln(2)+\frac{3}{2}\ln(2)-\frac{1}{2}\int_0^1 \ln(1-x)\ln(1+x^2)\,dx+2=$$

$$=\ln^2(2)+\frac{1}{2}\ln(2)+\frac{1}{4}\pi\ln(2)-\frac{1}{2}\int_0^1 \ln(1-x)\ln(1+x^2)\,dx+2$$

$$J=\int_0^1 \ln(1-x)\ln(1+x^2)\,dx=-\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\int_0^1 x^{2n}\ln(1-x)\,dx=$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\frac{H_{2n+1}}{2n+1}=\sum_{n=1}^{\infty}(-1)^n H_{2n+1}\left(\frac{1}{n}-\frac{2}{2n+1}\right)=$$

$$-\sum_{n=1}^{\infty}(-1)^n\left(H_{2n}+\frac{1}{2n+1}\right)\left(\frac{1}{n}-\frac{2}{2n+1}\right)=\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{n}-\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n+1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n}-2\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n+1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2\Re\sum_{n=1}^{\infty}i^n\frac{H_n}{n}-2\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n+1}x^{2n+1}]_{x=1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2\Re(Li_2(i)+\frac{1}{2}\ln^2(1-i))+\tan^{-1}(x)\ln(1+x^2)]_{x=1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2(-\frac{5\pi^2}{96}+\frac{1}{8}\ln^2(2))+\frac{\pi}{4}\ln(2)-2G-\ln(2)-\frac{\pi}{2}+4=$$

$$-\frac{5\pi^2}{48}+\frac{1}{4}\ln^2(2)+\frac{\pi}{4}\ln(2)-2G-\ln(2)-\frac{\pi}{2}+4$$

$$I=\ln^2(2)+\frac{1}{2}\ln(2)+\frac{1}{4}\pi\ln(2)+2-\frac{1}{2}(-\frac{5\pi^2}{48}+\frac{1}{4}\ln^2(2)+\frac{\pi}{4}\ln(2)-2G-\ln(2)-\frac{\pi}{2}+4)$$

$$=\frac{7}{8}\ln^2(2)+G+\frac{5\pi^2}{96}+\frac{\pi}{8}(2+\ln(2))+\ln(2)$$

Note : G --> Catalan's constant

2351. Find:

$$\int_0^1\int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Gbenga Ajeigbe-Nigeria

$$I=\int_0^1\int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}dxdy$$

$$=\int_0^1\int_0^1 \frac{\ln(x)\ln(1+x^2)}{x(1+y^2)^2}dxdy+\int_0^1\int_0^1 \frac{\ln(y)\ln(1+x^2)}{x(1+y^2)^2}dxdy$$

$$I=\int_0^1 \frac{dy}{(1+y^2)^2}\int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}dx+\int_0^1 \frac{\ln(y)}{(1+y^2)^2}dy\int_0^1 \frac{\ln(1+x^2)}{x}dx$$

$$I=\frac{1}{2}[\frac{y}{y^2+1}+\arctan(x)]_0^1\left[-\frac{3\zeta(3)}{4}\right]+([-\frac{G}{2}-\frac{1}{8}][-\frac{1}{2}Li_2(\text{-}1)]$$

$$[\frac{1}{4}+\frac{\pi}{8}]\left[-\frac{3\zeta(3)}{4}\right]+\left[-\frac{G}{2}-\frac{\pi}{8}\right][\frac{\pi^2}{24}]$$

$$I=-\frac{3\zeta(3)}{16}-\frac{3\zeta(3)}{32}\pi-\pi^2\frac{G}{48}\text{-}\frac{\pi^3}{192}$$

$$I=\frac{1}{384}[-8\pi^2G-9\pi\zeta(3)-18\zeta(3)-2\pi^3]$$

Solution 2 by Kartick Chandra Betal-India

$$\int_0^1\int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}dxdy$$

$$=\int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}dx\int_0^1 \frac{dy}{(1+y^2)^2}$$

$$+\int_0^1 \frac{\ln(1+x^2)}{x}dx\int_0^1 \frac{\ln(y)}{(1+y^2)^2}dy=AB+CD=\left(-\frac{3}{16}\zeta(3)\right)\left(\frac{\pi+2}{8}\right)+\left\{-\left(\frac{G}{2}+\frac{\pi}{8}\right)\right\}=$$

$$\frac{1}{384}\{-9\pi\zeta(3)-18\zeta(3)-8G\pi^2-2\pi^3\}$$

$$A=\int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}dx$$

$$=\frac{1}{4}\int_0^1 \frac{\ln(x)\ln(!+x)}{x}dx=\frac{1}{4}[-Li_2(-x)\ln(x)]_0^1+\frac{1}{4}[Li_3(-x)]_0^1=-\frac{3}{16}\zeta(3)$$

$$B=\int_0^1 \frac{dy}{(1+y^2)^2}=\int_0^{\frac{\pi}{4}} cos^2(y)dy=\frac{1}{2}[y+\frac{\sin(2y)}{2}]_0^{\frac{\pi}{4}}=(\frac{\pi+2}{8})$$

$$C=\int_0^1 \frac{\ln(1+x^2)}{x}dx=\frac{1}{2}\int_0^1 \frac{\ln(1+x)}{x}dx=\frac{\pi^2}{24}$$

$$D=\int_0^1\frac{\ln(y)}{(1+y^2)^2}dy=\frac{1}{2}\left[\left(\arctan(y)+\frac{y}{1+y^2}\right)\ln(y)\right]_0^1-\frac{1}{2}\int_0^1\frac{\left(\arctan(y)+\frac{y}{1+y^2}\right)}{y}dy$$
$$=-\frac{1}{2}\int_0^1\frac{\arctan(y)}{y}dy-\frac{1}{2}[arctan(y)]_0^1$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln(\tan(y))\,dy-\frac{\pi}{8}=-\left(\frac{G}{2}+\frac{\pi}{8}\right)$$

Note : $\begin{cases}G\to is\ Catalan's\ constant\\ \zeta(3)\to is\ Apery's\ constant\end{cases}$

2352. Find:

$$I=\int_0^\infty\frac{\tan^{-1}(x)}{x(1+x)^3}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty\frac{\tan^{-1}(x)}{x(1+x)^3}dx=\int_0^1\frac{\tan^{-1}(x)}{x(1+x)^3}dx+\int_1^\infty\frac{\tan^{-1}(x)}{x(1+x)^3}dx\Bigg]_{x\to1/x}$$
$$=\int_0^1\frac{\tan^{-1}(x)}{x(1+x)^3}dx+\int_0^1\frac{x^2\tan^{-1}\left(\frac{1}{x}\right)}{(1+x)^3}dx$$
$$=\int_0^1\frac{\tan^{-1}(x)}{x(1+x)^3}dx-\int_0^1\frac{x^2\tan^{-1}(x)}{(1+x)^3}dx+\frac{\pi}{2}\int_0^1\frac{x^2}{(1+x)^3}dx$$
$$=\int_0^1\tan^{-1}(x)\left\{\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right\}dx$$
$$-\int_0^1\tan^{-1}(x)\left\{\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right\}dx+\frac{\pi}{2}\int_0^1\frac{x^2}{(1+x)^3}dx$$
$$=\int_0^1\frac{\tan^{-1}(x)}{x}dx-2\int_0^1\frac{\tan^{-1}(x)}{x+1}dx+\int_0^1\frac{\tan^{-1}(x)}{(x+1)^2}dx-2\int_0^1\frac{\tan^{-1}(x)}{(x+1)^3}dx$$
$$+\frac{\pi}{2}\int_0^1\frac{x^2}{(1+x)^3}dx$$
$$=G-2\left\{\frac{\pi}{8}\log(2)\right\}+\left\{\frac{\log(2)}{4}\right\}-2\left\{\frac{\log(2)}{8}-\frac{\pi}{32}+\frac{1}{8}\right\}+\frac{\pi}{2}\left\{\log(2)-\frac{5}{8}\right\}$$
$$=G+\frac{\pi}{4}\log(2)-\frac{\pi}{4}-\frac{1}{4}$$

2353. If $0<a\le b\le\pi$ then:

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx \le \frac{2(b-a)}{ab}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban-Romania

$$\left(\frac{\cos x}{x}\right)' = \frac{-x\sin x - \cos x}{x^2} = -\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx = -\frac{\cos x}{x}\Big|_a^b =$$

$$= -\frac{\cos b}{b} + \frac{\cos a}{a} = \frac{b\cos a - a\cos b}{ab} \le \frac{2(b-a)}{ab}, ab > 0 \Rightarrow b\cos a - a\cos b$$

$$\le 2b - 2a \Rightarrow 2a - a\cos b \le 2b - b\cdot\cos a$$

$$a(2-\cos b) \le b(2-\cos a), 2-\cos \alpha > 0$$

because $\cos\alpha \le 1 < 2 \Rightarrow 2 - \cos\alpha > 0 \Rightarrow \frac{2-\cos b}{b} \le \frac{2-\cos a}{a}, f:(0,\pi] \to \mathbb{R}$

$$f(x) = \frac{2-\cos x}{x}, f(b) \le f(a), a \le b$$

$$f'(x) = \frac{x\sin x - 2 + \cos x}{x^2}; \varphi:(0,\pi] \to \mathbb{R}$$

$$\varphi(x) = x\sin x - 2 + \cos x, \varphi'(x) = \sin x + x\cos x - \sin x, \varphi'(x) = x\cos x,$$

$$\varphi'(x) = 0 \Rightarrow x = \frac{\pi}{2}$$

x	0	$\frac{\pi}{2}$	π
$f'(x)$	$+++++++$	0	$------$
$f(x)$	-1 ↗	$\frac{\pi}{2} - 2 < 0$	↘ -3

$$\Rightarrow \varphi(x) < 0; (\forall)x \in (0,\pi] \Rightarrow f'(x) < 0, (\forall)x \in (0,\pi] \Rightarrow f \searrow (0,\pi]$$

$a \le b \Rightarrow f(a) \ge f(b)$, true.

Then

$$\int_a^b \frac{\sin x}{x}dx + \int_a^b \frac{\cos x}{x^2}dx \le \frac{2(b-a)}{ab}, \qquad (\forall)0 < a \le b \le \pi$$

Solution 2 by Adrian Popa-Romania

$$\frac{2(b-a)}{ab}=2\left(\frac{1}{a}-\frac{1}{b}\right)=2\left.\frac{1}{x}\right|_a^b=2\int_a^b\frac{1}{x^2}dx$$

So we must show that $x\sin x+\cos x\le 2;(\forall)x\in(0,\pi)$

$$f(x)=x\sin x+\cos x-2$$

$$f'(x)=\sin x+x\cos x-\sin x=x\cos x=0\Rightarrow\begin{matrix}x=0\\x=\frac{\pi}{2}\end{matrix}$$

x	0 $\qquad\frac{\pi}{2}\qquad$ π
$f'(x)$	$+++++++0------$
$f(x)$	$-1\nearrow\frac{\pi}{2}-2<0\searrow-3$

$$\Rightarrow f(x)<0\ \ (\forall)x\in[0,\pi)\Rightarrow$$

$$\Rightarrow x\sin x+\cos x<2\Rightarrow\int_a^b\frac{\sin x}{x}dx+\int_a^b\frac{\cos x}{x^2}dx\le\frac{2(b-a)}{ab}$$

Solution 3 by Khaled Abd Imouti-Syria

$$\underbrace{\int_a^b\frac{\sin x}{x}dx}_{I_1}+\underbrace{\int_a^b\frac{\cos x}{x^2}dx}_{I_2}\le\frac{2(b-a)}{a\cdot b}\quad\text{(I)}$$

$$I_1=\int_a^b\frac{1}{x}\sin x\,dx\quad\left(\begin{matrix}u(x)=\frac{1}{x}\Rightarrow du=-\frac{1}{x^2}dx\\dv=\sin dx\Rightarrow v(x)=-\cos x\end{matrix}\right)$$

by using integration by parts:

$$I_1=\left[-\frac{1}{x}\cos x\right]_a^b-\int_a^b\frac{\cos x}{x^2}dx,I_1+I_2=\left[-\frac{1}{b}\cos(b)+\frac{1}{a}\cos(a)\right]$$

$$I_1+I_2=\frac{b\cos(a)-a\cdot\cos(b)}{a\cdot b}\quad(*)$$

Now let's prove: $b\cos a-a\cdot\cos(b)\overset{?}{\le}(b-a)\cdot 2$

$$b\cdot\cos a-2b-a\cdot\cos b+2a\overset{?}{\le}0,\qquad b(\cos a-2)+a(2-\cos b)\overset{?}{\le}0$$

$$b(\cos a-2)\overset{?}{\le}a\cdot(\cos b-2),\qquad\frac{\cos a-2}{a}\overset{?}{\le}\frac{\cos b-2}{b}$$

Suppose $f(x)=\frac{\cos x-2}{x},x\in]0,\pi],f'(x)=\frac{-x\sin x-\cos x+2}{x^2}=\frac{g(x)}{x^2}$

$$g(x) = -x \cdot \sin x - \cos x + 2, 'g(x) = -(\sin x + x \cdot \cos x) - (\sin x)$$

$$'g(x) = -x \cdot \cos x, 'g(x) = 0 \Rightarrow x = \frac{\pi}{2}$$

x	0 $\quad\quad$ $\frac{\pi}{2}$ $\quad\quad$ π
$'g(x)$	$- - - - - - - - 0 + + + + + + +$
$g(x)$	$\searrow$ $\frac{4-\pi}{2}$ $\nearrow$

So: $g(x) > 0$

$'f(x) > 0, f$ is completely increasing on $[0, \pi]$ so: $a \le b \Rightarrow f(a) \le f(b)$

$\frac{\cos a - 2}{a} a \le \frac{\cos b - 2}{b}$ and then: $b \cdot \cos a - a \cdot \cos(b) \le (b - a)$

and hence: $I_1 + I_2 \le \frac{2(b-a)}{a \cdot b}$

Solution 4 by Hikmat Mammadov – Azerbaijan

We have

$$\int_a^b \frac{\sin(x)}{x} dx + \int_a^b \frac{\cos(x)}{x^2} = \int_a^b \frac{d}{dx}\left(-\frac{\cos(x)}{x}\right) dx = \left[-\frac{\cos(x)}{x}\right]_a^b$$

$$= \frac{\cos(a)}{a} - \frac{\cos(b)}{b} = d\cos\left(\frac{1}{d}\right) - c\cos\left(\frac{1}{c}\right)$$

By noting $d = \frac{1}{a}$ and $c = \frac{1}{b}$ (we have $\frac{1}{\pi} \le c \le d$)

The derivatives of the function $f: \to x\cos\left(\frac{1}{x}\right)$ on $\left[\frac{1}{\pi}, +\infty\right[$

Are $f': x \to \cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right)$ and $f'': x \to -\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$

Since $f''(x) \ge 0$ if $x \in \left[\frac{1}{\pi}, \frac{2}{\pi}\right]$ and $f''(x) \le 0$ if $x \in \left[\frac{2}{\pi}, +\infty\right[$

The maximum of f' on $\left[\frac{1}{\pi} + \infty\right[$ is worth $f'\left(\frac{2}{\pi}\right) = \frac{\pi}{2}$

We know that there exists $\alpha \in]c, d[$ such that $f(d) - f(c) = f'(\alpha)(d - c)$

So $f(d) - f(c) \le \frac{\pi}{2}(d - c)$ so $\frac{\cos(a)}{a} - \frac{\cos(b)}{b} \le \frac{\pi}{2}\left(\frac{1}{a} - \frac{1}{b}\right)$

Finally

$$\int_a^b \frac{\sin(x)}{x} dx + \int_a^b \frac{\cos(x)}{x^2} dx \le \frac{\pi}{2}\frac{b-a}{ab}$$

$$\int_a^b \frac{\sin(x)}{x}dx + \int_a^b \frac{\cos(x)}{x^2}dx \leq \frac{2(b-a)}{ab}$$

2354. Find:

$$\Omega = \int_0^1 \int_0^1 \ln\left(\sqrt{\frac{x}{x^2+1} + \frac{y}{y^2+1}}\right) dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \frac{1}{2}\int_0^1 \int_0^1 \ln\left(\frac{x}{1+x^2} + \frac{y}{y^2+1}\right) dxdx = \frac{1}{2}\int_0^1 \int_0^1 \left(\frac{x(1+y^2)+y(1+x^2)}{(1+x^2)(1+y^2)}\right) dxdy = \frac{1}{2}(\Omega_1 - \Omega_2)$$

$$\Omega_1 = \int_0^1 \int_0^1 \ln\left(x(1+y^2)+y(1+x^2)\right) dxdy = \int_0^1 \int_0^1 \ln(x+xy^2+y+x^2y)\, dxdy =$$

$$\int_0^1 \int_0^1 \ln((xy+1)(x+y))\, dxdy = \int_0^1 \int_0^1 \ln(1+xy)\, dxdy + \int_0^1 \int_0^1 \ln(x+y)\, dxdy =$$

$$\underbrace{-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 \int_0^1 x^n y^n\, dxdy}_{} + \underbrace{\int_0^1 \int_0^1 \ln(x+y)\, dxdy}_{} = I_1 + I_2$$

$$I_1 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)^2} = -\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}\right) = -2 + 2\ln(2) + \frac{\pi^2}{12}$$

$$I_2 = \int_0^1 \int_0^1 \ln(x+y)\, dxdy = \int_0^1 \left((x+y)\ln(x+y) - x\right)\Big|_0^1 dy =$$

$$\int_0^1 \left((1+y)\ln(1+y) - y\ln(y) - 1\right)dy = \frac{1}{2}\left[(y+1)^2\ln(1+y) - 3y - y^2\ln(y)\right]_0^1 =$$

ln(4)-1.5 $\Omega_1 = I_1 + I_2 = \ln(16) + \frac{\pi^2}{12} - 3.5$

$$\Omega_2 = \int_0^1 \int_0^1 \ln(1+x^2)\, dxdy + \int_0^1 \int_0^1 \ln(1+y^2)\, dxdy = 2\int_0^1 \int_0^1 \ln(1+x^2)\, dxdy =$$

$$-2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 \int_0^1 x^{2n} dxdy = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} =$$

$=2\ln(2)+\pi-4$

$$\Omega=\frac{1}{2}(\Omega_1-\Omega_2)=\frac{1}{4}+\frac{\pi^2}{24}+\ln(2)-\frac{\pi}{2}$$

2355. Find:

$$\Omega=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(1+x^2)\,dx\overset{IBP}{\cong}$$

$$\sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{2n+1}}{2n+1}\ln(1+x^2)\right]_0^1-2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1\frac{x^{2n+1}}{x^2+1}dx=$$

$$\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}-2\int_0^1\frac{x\tan^{-1}}{(x^2+1)}dx=\ln(2)\tan^{-1}(1)-2\int_0^1\frac{x\tan^{-1}(x)}{(1+x^2)}dx=$$

$$\frac{\pi}{4}\ln(2)-2\int_0^1\frac{x\tan^{-1}(x)}{1+x^2}dx\overset{\arctan(x)=>t}{\cong}\frac{\pi}{4}\ln(2)-2\int_0^{\frac{\pi}{4}}t\tan(t)dt=$$

$$\frac{\pi}{4}\ln(2)+2[t\log(\cos(t))]_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt=2\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt=$$

$$2\ln(2)\int_0^{\frac{\pi}{4}}dt+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nt)\,dt=\frac{\pi}{2}\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\frac{\pi n}{2})}{n^2}=$$
$$\frac{\pi}{2}\ln(2)-G$$

$$\Omega=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx=\frac{\pi}{2}\ln(2)-G$$

Solution 2 by Bui Hong Suc-Vietnam

We have :

$$\begin{Bmatrix}\int_0^{\frac{\pi}{4}}\ln(\tan(x))\,dx=\int_0^{\frac{\pi}{4}}\ln(\sin(x))\,dx-\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=-G\\ \int_0^{\frac{\pi}{2}}\ln(\sin(x))\,dx=\int_0^{\frac{\pi}{4}}\ln(\sin(x))\,dx+\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=-\frac{\pi}{2}\ln(2)\end{Bmatrix}$$

$$\Omega=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx\underbrace{=}_{x=\tan(t)}\int_0^{\frac{\pi}{4}}\frac{\ln(\frac{1}{\cos^2(t)})}{\frac{1}{\cos^2(t)}}\frac{dt}{\cos^2(t)}=-2\int_0^{\frac{\pi}{4}}\ln(\cos(t))=$$

$$-2(-\frac{\pi}{4}\ln(2)+\frac{G}{2})=\frac{\pi}{2}\ln(2)-G$$

2356. Find:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-x^2y^2)+xyln(\frac{1-xy}{1+xy})}{1-x^2y^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Bui Hong Suc-Vietnam

We have: $\zeta(s)\Gamma(s)=\int_0^\infty\frac{x^{z-1}}{e^x-1}dx\ \ Li_3(1)=\zeta(3)$

$$Li_3\left(\frac{1}{2}\right)=\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7\zeta(3)}{8},Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2(2)}{2}$$

$$\Omega=\frac{\ln(1-x^2y^2)+xyln(\frac{1-xy}{1+xy})}{1-x^2y^2}dxdy=$$

$$\int_0^1\int_0^1\frac{\ln(1-xy)+\ln(1+xy)+xy(\ln(1-xy)-\ln(1+xy))}{(1-xy)(1+xy)}dxdy=$$

$$\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}dxdy+\int_0^1\int_0^1\frac{\ln(1+xy)}{1+xy}dxdy=-\int_0^1\frac{dy}{y}\int_0^1\ln(1-xy)\,d(\ln(1-xy))+$$

$$\int_0^1\frac{dy}{y}\int_0^1\ln(1+xy)\,d(\ln(1+xy))=\frac{1}{2}(-\int_0^1\frac{dy}{y}ln^2(1-xy)\Big|_0^1+$$

$$\int_0^1\frac{dy}{y}ln^2(1+xy)\Big|_0^1)=\frac{1}{2}(-\underbrace{\int_0^1\frac{ln^2(1-y)dy}{y}}_{I}+\underbrace{\int_0^1\frac{ln^2(1+y)}{y}dy}_{J}$$

$$I=\int_0^1\frac{ln^2(1-y)dy}{y}\overset{\ln(1-y)=-x}{\cong}\int_0^\infty\frac{x^2e^{-x}}{1-e^{-x}}=\int_0^\infty\frac{x^2dx}{e^x-1}=\zeta(3)\Gamma(3)=2\zeta(3)$$

$$J=\int_0^1\frac{ln^2(1+y)}{y}dy\overset{X=\frac{1}{y+1}}{\cong}\int_{\frac{1}{2}}^1\frac{ln^2(x)}{\frac{1-x}{x}x^2}dx=\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{ln^2(x)}{1-x}dx+\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x}dx=$$

$$(-ln^2(x)\ln(1-x)-2\ln(x)Li_2(x)+2Li_3(x))\Big|_{\frac{1}{2}}^1+\frac{ln^3(x)}{3}\Big|_{\frac{1}{2}}^1=$$

$$2Li_3(1)-2Li_3\left(\frac{1}{2}\right)-2\ln(2)Li_2\left(\frac{1}{2}\right)-ln^3(2)+\frac{ln^3(2)}{2}=2\zeta(3)-$$

$$2\left\{\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7\zeta(3)}{8}\right\}-2\ln(2)\left\{\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right\}-\frac{2ln^3(2)}{3}=\frac{\zeta(3)}{4}$$

Hence: $\Omega=\frac{1}{2}\left\{-2\zeta(3)+\frac{\zeta(3)}{4}\right\}=-\frac{7}{8}\zeta(3)$

Note: ζ(3)=Apery's constant

2357. Find:

$$\int_0^1 \frac{xln^2(x)}{x^2+x+1}dx$$

Proposed by Shirvan Tahirov-AZerbaijan

Solution 1 by Pham Duc Nam-Vietnam

We have: $\sum_{n=1}^{\infty}\sin(n\theta)\,x^n=\frac{x\sin(\theta)}{x^2-2x\cos(\theta)+1}$

=>Let: $\theta=\frac{2\pi}{3}$ => $\frac{x}{x^2+x+1}=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{2\pi n}{3}\right)x^n$

=>$\Omega=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{2\pi n}{3}\right)\int_0^1 x^n ln^2(x)dx=\frac{4}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{2\pi n}{3}\right)\frac{1}{(n+1)^3}$

And we also have:

$$\text{Sin}\left(\frac{2\pi n}{3}\right)=\begin{cases}0, if\ n=0\ (mod\ 3)\\ \frac{\sqrt{3}}{2}, if\ n=1(mod3)\\ -\frac{\sqrt{3}}{2}, if\ n=2(mod3)\end{cases}$$

$$>=\Omega=2\sum_{n=0}^{\infty}\frac{1}{(3k+2)^3}-2\sum_{n=0}^{\infty}\frac{1}{(3k+3)^3}=2\left(\frac{13}{27}\zeta(3)-\frac{2}{81\sqrt{3}}\pi^3\right)-\frac{2}{27}\zeta(3)=\frac{8}{9}\zeta(3)-\frac{4}{81\sqrt{3}}$$

Note: $\psi^{(2)}\left(\frac{2}{3}\right)=\frac{\pi}{2}\frac{d^2}{dz^2}\cot(\pi z)]_{z=\frac{1}{3}}-(2!)9\left(\zeta(3)+Cl_3\left(\frac{2\pi}{3}\right)\right)$

And : $Cl_{2k+1}\left(\frac{2\pi}{3}\right)=\frac{1}{2}(1-3^{-2k})\zeta(2k+1)$

$$=>\psi^{(2)}\left(\frac{2}{3}\right)=\frac{4}{3\sqrt{3}}\pi^3-18\left(\zeta(3)+\frac{8}{18}\zeta(3)\right)=\frac{4}{3\sqrt{3}}\pi^3-26\zeta(3)$$

$$\text{But:}\sum_{k=0}^{\infty}\frac{1}{(3k+2)^3}=-\frac{1}{54}\left(\frac{4}{3\sqrt{3}}\pi^3-26\zeta(3)\right)=\frac{13}{27}\zeta(3)-\frac{2}{81\sqrt{3}}\pi^3$$

Solution 2 by Bui Hong Suc-Vietnam

$$\int_0^1 x^n ln^2(x)dx=x^{n+1}\left[\frac{ln^2(x)}{n+1}-\frac{2\ln(x)}{(n+1)^2}+\frac{2}{(n+1)^3}\right]$$

$$\zeta\left(3,\tfrac{2}{3}\right)=\sum_{n=0}^{\infty}\frac{1}{(n+\frac{2}{3})^3}=13\zeta(3)-\frac{2\pi^3}{3\sqrt{3}}$$

$$\Omega_{n,k}=\int_0^1\frac{x^k ln^2(x)}{\sum_{i=0}^{n-1}x^i}dx=\int_0^1\frac{x^k(1-x)ln^2(x)dx}{(1-x)\sum_{i=0}^{n-1}x^i}=\int_0^1\frac{(x^k-x^{k+1})ln^2(x)}{1-x^n}dx=$$

$$\sum_{j=0}^{\infty}\int_0^1 (x^k - x^{k+1})x^{nj} ln^2(x)dx = \sum_{j=0}^{\infty}\left\{\int_0^1 x^{nj+k} ln^2(x)dx - \int_0^1 x^{nj+k+1} ln^2(x)dx\right\} =$$

$$\sum_{j=0}^{\infty}\left\{\begin{matrix} x^{nj+k+1}\left[\frac{ln^2(x)}{nj+k+1} - \frac{2\ln(x)}{(nj+k+1)^2} + \frac{2}{(nj+k+1)^3}\right] \\ -x^{nj+k+2}\left[\frac{ln^2(x)}{nj+k+2} - \frac{2\ln(x)}{(nj+k+2)^2} + \frac{2}{(nj+k+2)^3}\right]\end{matrix}\right\}\Bigg|_0^1$$

$$= \sum_{j=0}^{\infty}\left\{\frac{2}{(nj+k+1)^3} - \frac{2}{(nj+k+2)^3}\right\} = \frac{2}{n^3}\sum_{j=0}^{\infty}\left\{\frac{1}{(j+\frac{k+1}{n})^3} - \frac{1}{(j+\frac{k+2}{n})^3}\right\} =$$

$\frac{2}{n^3}\left(\zeta\left(3,\frac{k+1}{n}\right) - \zeta\left(3,\frac{k+2}{n}\right)\right)$. As : n=3,k=1 $\Omega = \int_0^1 \frac{x ln^2(x)}{x^2+x+1}dx = \frac{2}{3^3}(\zeta\left(3,\frac{2}{3}\right) - \zeta(3,\frac{3}{3})) =$

$$\frac{2}{27}\left(13\zeta(3) - \frac{2\pi^3}{3\sqrt{3}} - \zeta(3)\right) = \frac{8}{9}\zeta(3) - \frac{4\pi^3}{81\sqrt{3}}$$

2358. Prove that:

$$I = \int_0^1\int_0^1\int_0^1 \sum_{x,y,z}\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz = \frac{128}{35}(2-\sqrt{2})$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^1\int_0^1\int_0^1 \left\{\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}} + \sqrt{\frac{\sqrt{y}}{\sqrt{x}+\sqrt{z}}} + \sqrt{\frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}}}\right\} dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 \sqrt{\frac{\sqrt{y}}{\sqrt{x}+\sqrt{z}}}\,dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \sqrt{\frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}}}\,dx\,dy\,dz = 3\int_0^1\int_0^1\int_0^1 \sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}\,dx\,dy\,dz$$

$$= \frac{12}{5}\int_0^1\int_0^1 \frac{1}{\sqrt{\sqrt{y}+\sqrt{z}}}\,dy\,dz$$

$$\text{Let } \sqrt{\sqrt{y}+\sqrt{z}} = t$$

$$I=\frac{12}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}(4t^2-4\sqrt{z})dt\,dz=\frac{48}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}t^2dt\,dz-\frac{48}{5}\int_0^1\int_{\sqrt{\sqrt{z}}}^{\sqrt{1+\sqrt{z}}}\sqrt{z}dt\,dz$$

$$=\frac{48}{5}\left\{\frac{16\sqrt{2}-4}{35}\right\}-\frac{48}{5}\left\{\frac{88\sqrt{2}-92}{105}\right\}=\frac{128}{35}(2-\sqrt{2})$$

2359. Find:

$$\Omega=\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

We have $\overbrace{\int_0^1 x\ln(\arccos^2(1-x^2))dx}^{x^2\to x}+\overbrace{\int_0^1\frac{\ln^2(1-x)}{x}dx}^{1-x\to}=$

$\overbrace{\int_0^1 \ln\cos^{-1}(1-x)dx}^{1-x\to}+\overbrace{\int_0^1\frac{\ln^2(x)}{1-x}}^{|x|<1}=\overbrace{\int_0^1 \ln\cos^{-1}(x)dx}^{\cos^{-1}(x)\to x}+\underbrace{\overbrace{\sum_{n\in N}\int_0^1 x^{n-1}\ln^2(x)dx}^{\text{Note section }(1)}}_{\text{Note section}(2)}=$

$\int_0^{\frac{\pi}{2}}\sin(x)\ln(x)\,dx+2\sum_{n\in N}\frac{1}{n^3}\overset{\frown}{=>}\ln(\frac{\pi}{2})-Cin(\frac{\pi}{2})+2\zeta(3)\quad\overset{\text{Note section }(3)}{=>}\quad=$

$$Ci(\tfrac{\pi}{2})-\gamma-\ln\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi}{2}\right)+2\zeta(3)$$

$$\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx=2\zeta(3)+Ci(\tfrac{\pi}{2})-\gamma$$

Note Section:

1) $\int_0^1 x^m\ln^n(x)dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$

2) $\int_0^x\frac{1-\cos(t)}{t}dt=Cin(x)$

3) $Ci(x)=\ln(x)-Cin(x)+\gamma$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1 x\ln\left(\arccos^2(1-x^2)\right)dx\overset{1-x^2\to x}{\triangleq}$$

$$\Omega_1=\int_0^1\log(\arccos(x))\,dx\to\left\{\arccos(x)=t;\ \frac{dt}{dx}=\frac{1}{\sin(t)}\right\}=$$

$$\Omega_1 = \int_0^{\frac{\pi}{2}} \sin(t)\ln(t)\,dt \overset{IBP}{=} -\Big|_0^{\frac{\pi}{2}} \cos(t)\ln(t) + \int_0^{\frac{\pi}{2}} \frac{\cos(t)}{t}\,dt =$$

$$\frac{\pi}{2}\ln(0) + \int_0^{\frac{\pi}{2}} \frac{\cos(t)}{t}\,dt = -\int_0^{\frac{\pi}{2}} \frac{1}{t}\,dt + \int_0^{\frac{\pi}{2}} \frac{\cos(t)}{t}\,dt + \ln\left(\frac{\pi}{2}\right) =$$

$$\ln\left(\frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} \frac{\cos(t)-1}{t}\,dt = Ci\left(\frac{\pi}{2}\right) - \gamma$$

$$\Omega_2 = \int_0^1 \frac{ln^2(1-x)}{x}\,dx \overset{1-x\to x}{=} \int_0^1 \frac{ln^2(x)}{1-x}\,dx = \sum_{n=0}^{\infty} \int_0^1 x^n ln^2(x)\,dx \overset{IBP}{=} 2\zeta(3)$$

$$\Omega = \Omega_1 + \Omega_2 = 2\zeta(3) + Ci\left(\frac{\pi}{2}\right) - \gamma$$

Note section: Cosine integral $Ci(z) = \ln(z) + \gamma + \int_0^z \frac{\cos(x)-1}{x}\,dx$

For $a \in Z$, the following identity holds: $\int_0^1 \frac{ln^a(x)}{1-x}\,dx = (-1)^a \Gamma(a+1)\zeta(a+1)$

2360. Find:

$$\Omega = \int_0^1 \frac{\ln(x^2+1) + \arctan(x)}{x^2+1}\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 \frac{\ln(x^2+1) + \arctan(x)}{x^2+1}\,dx \overset{\arctan(x)\to x}{=} \int_0^{\frac{\pi}{4}} \left[\ln\left(1 + tan^2(x)\right) + x\right] dx$$

$$= \left(\frac{x^2}{2}\right)\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln\left(sec^2(x)\right) dx =$$

$$\Omega = \frac{1}{2}\left(\frac{\pi^2}{16} - 0\right) - 2\left[-\int_0^{\frac{\pi}{4}} \ln(2)\,dx - \sum_{n\in N} \frac{(-1)^n}{n} \int_0^{\frac{\pi}{4}} \cos(2nx)\,dx\right] =$$

$$\frac{\pi^2}{32} + 2\left[\frac{\pi}{4}\ln(2) + \sum_{n\in N} \frac{(-1)^n}{n} \left(\frac{\sin(2nx)}{2n}\right)\Big|_0^{\frac{\pi}{4}}\right] =$$

$$\frac{\pi^2}{32} + \frac{\pi}{2}\ln(2) + \sum_{n\in N} \frac{(-1)^n}{n} \sin\left(\frac{\pi n}{2}\right) = \frac{\pi^2}{32} + \frac{\pi}{2}\ln(2) - \sum_{n\in N} \frac{(-1)^n}{(2n+1)^2} =$$

$$= \frac{\pi^2}{32} + \frac{16\pi}{32}\ln(2) - G$$

Note section:

$$\begin{cases} \ln(\cos(x)) = -\ln(2) - \sum_{n\in N} \frac{(-1)^n}{n}\cos(2nx) \\ sin^2(x) + cos^2(x) = 1 \to 1 + tan^2(x) = sec^2(x) \\ Dirichlet\ Beta\ Function: \beta(2) \\ \beta(2) = \sum_{n\in N} \frac{(-1)^n}{(2n+1)^2} = G\ (Catalan'sconstant) \end{cases}$$

2361. Find:

$$\Omega = \int_1^\infty \frac{x\ln(x)(1-\ln(x))^2}{(1+x^2)(1+x)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega = \int_1^\infty \frac{x\ln(x)(1-\ln(x))^2}{(1+x)^2(1+x^2)}dx = \frac{1}{2}\underbrace{\int_1^\infty \frac{\ln(x)(1-\ln(x))^2}{(1+x^2)}dx}_{\Omega_1} - \frac{1}{2}\underbrace{\int_1^\infty \frac{\ln(x)(1-\ln(x))^2}{(1+x)^2}dx}_{\Omega_2}$$

$$\Omega_1 = \int_1^\infty \frac{\ln(x) - 2ln^2(2) + ln^3(x)}{1+x^2}dx = -\int_0^1 \frac{\ln(x) + 2ln^2(x) + ln^3(x)}{1+x^2}dx =$$

$$-\sum_{n=0}^\infty (-1)^n \int_0^1 x^{2n}(\ln(x) + 2ln^2(x) + ln^3(x))dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} - 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} +$$

$$6\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^4} = G - \frac{\pi^3}{8} + \frac{6}{i}\sum_{n=0}^\infty \frac{i^{2n+1}}{(2n+1)^4} \underbrace{\overset{\{\sum_{n=0}^\infty a_{2n+1} = \frac{1}{2}\sum_{n=1}^\infty a_n - \frac{1}{2}\sum_{n=1}^\infty (-1)^n a_{2n+1}\}}{=}}_{converting\ the\ summand}$$

$$G - \frac{\pi^3}{8} + \frac{3}{i}\sum_{n=1}^\infty \frac{i^n}{n^4} - \frac{3}{i}\sum_{n=1}^\infty \frac{(-1)^n i^n}{n^4} = G - \frac{\pi^3}{8} - 3iLi_4(i) + 3iLi_4(-i)$$

$$\Omega_2 = \int_1^\infty \frac{\ln(x)(1-\ln(x))^2}{(1+x)^2}dx$$

$$= -\int_0^1 \frac{\ln(x)(1+\ln(x))^2}{(1+x)^2}dx = -\int_0^1 \frac{\ln(x) + 2ln^2(x) + ln^3(x)}{(1+x)^2}dx =$$

$$\Omega_2 = \sum_{n=1}^\infty (-1)^n n\int_0^1 x^{n-1}\left(\ln(x) + 2ln^2(x) + ln^3(x)\right)dx \overset{\left\{\int_0^1 x^{n-1}ln^k(x)dx = \frac{(-1)^k k!}{n^{k+1}}\right\}}{=}$$

$$-\sum_{n=1}^\infty \frac{(-1)^n n}{n^2} + 4\sum_{n=1}^\infty \frac{(-1)^n}{n^3} - 6\sum_{n=1}^\infty \frac{(-1)^n n}{n^4} = \ln(2) - 4\eta(2) + 6\eta(3) \underbrace{\overset{\{\eta(z) = (1-2^{1-z})\zeta(z)\}}{=}}_{Dirichlet\ eta\ function}$$

$$\ln(2) - 2\zeta(2) + \frac{9}{2}\zeta(3) = \ln(2) + \frac{9}{2}\zeta(3) - \frac{\pi^2}{3}$$

$$\Omega = \frac{1}{2}(\Omega_1 - \Omega_2) = \frac{G}{2} + \frac{3}{2}iLi_4(-i) - \frac{3}{2}iLi_4(i) + \frac{\pi^2}{6} - \frac{\ln(2)}{2} - \frac{9}{4}\zeta(3) - \frac{\pi^3}{16}$$

Solution 2 by Gbenga Ajeigbe-Nigeria

$$I=\int_1^\infty \frac{xln(x)(1-\ln(x))^2}{(1+x^2)(1+x)^2}dx=\int_0^1 \frac{xln(\frac{1}{x})(1-\ln\left(\frac{1}{x}\right))^2}{(x^2+1)(x+1)^2}dx=\int_0^1 \frac{xln(\frac{1}{x})(1-2\ln\left(\frac{1}{x}\right)+ln^2\left(\frac{1}{x}\right))}{(x^2+1)(x+1)^2}dx=$$

$$I=\int_0^1 \frac{xln(\frac{1}{x})}{(x^2+1)(x+1)^2}dx-2\int_0^1 \frac{xln^2(\frac{1}{x})}{(x^2+1)(x+1)^2}dx+\int_0^1 \frac{xln^3(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=A-2B+C$$

$$A=\int_0^1 \frac{xln(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-\int_0^1 \frac{xln(x)}{(x^2+1)(x+1)^2}dx=\frac{1}{2}\int_0^1 \frac{x^2ln^2(x)}{x^2+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{(x+1)^2}dx$$

$$=-\frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+3)^2}+\frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2+n)^2}+\sum_{n=0}^\infty \frac{\binom{-2}{n}}{(2+n)^2}=-\frac{1}{2}(1-G)+\frac{1}{2}\left(1-\frac{\pi^2}{12}\right)+\frac{1}{2}\left(\frac{\pi^2}{12}-\ln(2)\right)=\frac{G}{2}-\frac{\ln(2)}{2}$$

$$B=-2\int_0^1 \frac{xln^2(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-2\int_0^1 \frac{xln^2(x)}{(x^2+1)(x+1)^2}dx=\int_0^1 \frac{x^2\ln(x)}{x^2+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{(x+1)^2}dx=2!\sum_{n=0}^\infty \frac{(-1)^n}{(3+2n)^3}-2!\sum_{n=0}^\infty \frac{(-1)^n}{(2+n)^3}-2!\sum_{n=0}^\infty \frac{\binom{-2}{n}}{(2+n)^3}=$$

$$2\left(\frac{1}{32}(32-\pi^3)\right)-2\left(\frac{1}{4}(4-3\zeta(3))\right)-2\left(\frac{1}{12}(9\zeta(3)-\pi^2)\right)=\frac{\pi^2}{6}-\frac{\pi^3}{16}$$

$$C=\int_0^1 \frac{xln^3(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-\int_0^1 \frac{xln^3(x)}{(x^2+1)(x+1)^2}dx=\frac{1}{2}\int_0^1 \frac{x^2ln^3(x)}{x^2+1}dx-$$

$$\frac{1}{2}\int_0^1 \frac{xln^3(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln^3(x)}{(x+1)^2}dx=-\frac{3!}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(3+2n)^4}+\frac{3!}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2+n)^4}+$$

$$\frac{3!}{2}\sum_{n=0}^\infty \frac{\binom{-2}{n}}{(2+n)^4}=\frac{3}{2}iLi_4(-i)-\frac{3}{2}iLi_4(i)-\frac{9\zeta(3)}{4}$$

$$I=A-2B+C$$

$$I=\frac{G}{2}+\frac{3}{2}iLi_4(-i)-\frac{3}{2}iLi_4(i)-\frac{9}{4}\zeta(3)-\frac{\ln(2)}{2}+\frac{\pi^2}{6}-\frac{\pi^3}{16}$$

Note : $\begin{cases} G\to Catalan's\ constant \\ \zeta(3)\to Apery's\ constant\end{cases}$

2362. Find:

$$I = \int_1^{\infty} \frac{x\ln(x)}{x^4 + x^2 + 1} dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_1^{\infty} \frac{x\ln(x)}{x^4 + x^2 + 1} dx \Bigg]_{x^2 \to 1/x} = -\frac{1}{4}\int_0^1 \frac{\ln(x)}{x^2 + x + 1} dx = -\frac{1}{4} I_1$$

$$I_1 = \int_0^1 \frac{\ln(x)}{x^2 + x + 1} dx = \int_0^1 \frac{(1-x)\ln(x)}{1 - x^3} dx = -\int_0^{\infty} \frac{y(1-e^{-y})e^{-y}}{1 - e^{-3y}} dy$$

$$= -\sum_{k=0}^{\infty} \int_0^{\infty} y(1-e^{-y})e^{-y}e^{-3ky}\, dy$$

$$= -\sum_{k=0}^{\infty} \int_0^{\infty} ye^{-(3k+1)y}\, dy + \sum_{k=0}^{\infty} \int_0^{\infty} ye^{-(3k+2)y}\, dy$$

$$= -\sum_{k=0}^{\infty} \frac{\Gamma(2)}{(3k+1)^2} + \sum_{k=0}^{\infty} \frac{\Gamma(2)}{(3k+2)^2}$$

$$= -\sum_{k=0}^{\infty} \frac{1}{(3k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(3k+2)^2} = -\frac{1}{9}\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2} + \frac{1}{9}\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{2}{3}\right)^2}$$

$$= -\frac{1}{9}\varphi'\left(\frac{1}{3}\right) + \frac{1}{9}\varphi'\left(\frac{2}{3}\right) = -\frac{1}{9}\varphi'\left(\frac{1}{3}\right) + \frac{1}{9}\left(\frac{4\pi^2}{3} - \varphi'\left(\frac{1}{3}\right)\right) = \frac{4\pi^2}{27} - \frac{2}{9}\varphi'\left(\frac{1}{3}\right)$$

$$= \frac{2}{9}\left(\frac{2\pi^2}{3} - \varphi'\left(\frac{1}{3}\right)\right)$$

$$I = -\frac{1}{4}I_1 = -\frac{1}{4}\left\{\frac{2}{9}\left(\frac{2\pi^2}{3} - \varphi'\left(\frac{1}{3}\right)\right)\right\} = \frac{1}{18}\varphi'\left(\frac{1}{3}\right) - \frac{\pi^2}{27}$$

2363. Find:

$$I = \lim_{n \to 0}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^k(x)} dx\right)^k\right)$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Togrul Ehmedov-Azerbaijan

We know that

$$I_1=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^k(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\cos^k(x)}{\sin^k(x)+\cos^k(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^k(x)}{\sin^k(x)+\cos^k(x)}dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^k(x)+\cos^k(x)}{\sin^k(x)+\cos^k(x)}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{4}$$

$$I=\lim_{n\to 0}\left(\frac{1}{n}\sum_{k=1}^{n}(I_1)^k\right)=\lim_{n\to 0}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\pi}{4}\right)^k\right)=\lim_{n\to 0}\left(\frac{1}{n}\left(\frac{\frac{\pi}{4}\left(\left(\frac{\pi}{4}\right)^n-1\right)}{\frac{\pi}{4}-1}\right)\right)$$

$$=\frac{\pi}{\pi-4}\lim_{n\to 0}\left(\frac{\left(\frac{\pi}{4}\right)^n-1}{n}\right)=\frac{\pi}{\pi-4}\lim_{n\to 0}\left(\left(\frac{\pi}{4}\right)^n\log\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{\pi-4}\log\left(\frac{\pi}{4}\right)$$

2364. Find:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n\psi^{(0)}(n)}{n^2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n(H_{n-\gamma}-1)}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^2}\left(H_n-\frac{1}{n}-\gamma\right)=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n^2}-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^3}-\gamma\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^2}=\Omega_1-\Omega_2-\gamma\Omega_3$$

$$\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n^2}\overset{\left(\overline{H}_n=\ln(2)-\int_0^1\frac{(-x)^n}{1+x}dx\right)}{=}\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}$$

$$-\sum_{n=1}^{\infty}\frac{H_n}{n^2}\int_0^1\frac{x^n}{1+x}dx$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\int_0^1\frac{Li_3(x)}{1+x}dx+\int_0^1\frac{Li_3(1-x)}{1+x}dx-\int_0^1\frac{\ln(1-x)Li_2(1-x)}{1+x}dx-$$

$$\frac{1}{2}\int_0^1\frac{\ln(x)ln^2(1+x)}{1+x}dx-\zeta(3)\int_0^1\frac{1}{1+x}dx=J_1\ln(2)-J_2+J_3-J_4-J_5-J_6\zeta(3)$$

$$J_1=\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}$$

$$=-\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}\ln(x)\,dx$$

$$=-\int_0^1\frac{\ln(x)}{x}\left(\sum_{n=1}^{\infty}(-1)^n H_n x^n\right)dx$$

$$=\int_0^1\frac{\ln(x)\ln(1+x)}{x(1+x)}dx=\int_0^1\frac{\ln(x)\ln(1+x)}{x}dx-\underbrace{\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx}_{IBP}=$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}\ln(x)\,dx-$$

$$\left[\frac{1}{2}\ln(x)ln^2(1+x)\right]_0^1+\frac{1}{2}\int_0^1\frac{ln^2(1+x)}{x}dx=-\frac{3\zeta(3)}{4}+\frac{1}{2}\int_0^1\frac{ln^2(1+x)}{x}dx=-\frac{5\zeta(3)}{8}$$

$$J_2=\int_0^1\frac{Li_3(x)}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n Li_3(x)dx=-\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n-1}Li_3(x)dx$$

$$\overset{\left\{\int_0^1 x^{n-1}Li_3(x)=\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right\}}{\cong}-\sum_{n=1}^{\infty}(-1)^n\left(\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right)=$$

$$-\zeta(3)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}=\zeta(3)\ln(2)-\frac{1}{2}\zeta^2(2)-\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$$

$$=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}ln^2(x)dx=\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1+x)ln^2(x)}{x(1+x)}dx}_{subtitation\to x=\frac{t}{1-t}}$$

$$=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{ln^2\left(\frac{t}{1-t}\right)\ln(1-t)}{t}dt=\zeta(3)\ln(2)-\frac{\pi^4}{72}-$$

$$\frac{1}{2}\Big(\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{t}dt+\int_0^{\frac{1}{2}}\frac{ln^3(1-t)}{t}dt}_{L}$$

$$-2\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}}_{P}\qquad L=\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{t}dt}_{IBP}+\int_0^{\frac{1}{2}}\frac{ln^3(t)}{t}dt=\left[\frac{1}{3}\ln(1-t)ln^3(t)\right]_0^{\frac{1}{2}}+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt+$$

$$\underbrace{\int_0^{\frac{1}{2}}\frac{ln^3(1-t)}{t}dt}_{1-t\to t}=\frac{1}{3}ln^4(2)+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt-\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt+\int_0^1\frac{ln^3(t)}{1-t}dt=$$

$$4Li_4\left(\frac{1}{2}\right)+\frac{7}{2}\zeta(3)ln(2)+\frac{2}{3}ln^4(2)-\frac{\pi^2}{6}ln^4(2)-\frac{\pi^4}{15}$$

$$P=\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt\overset{IBP}{\cong}\frac{1}{2}ln^4(2)+\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{1-t}dt}_{1-t\to t}=\frac{1}{2}ln^4(2)-\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt$$

$$=\frac{1}{2}ln^4(2)-\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt+\int_0^1\frac{ln^2(1-t)\ln(t)}{t}dt=\frac{1}{2}ln^4(2)+\underbrace{\int_0^1\frac{\ln(t)ln^2(1-t)}{t}dt}_{1-t=>t}-P$$

$$2P=\frac{1}{2}ln^4(2)+2\zeta(4)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{1}{2}ln^4(2)+2\zeta(4)-\frac{5}{2}\zeta(4)\quad P=\frac{1}{4}ln^4(2)-\frac{\pi^4}{360}$$

$$J_2=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}(L-2P)$$
$$=\frac{\pi^2}{12}ln^2(2)-\frac{1}{12}ln^4(2)-2Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{60}-\frac{3}{4}\zeta(3)\ln(2)$$

$$J_3=\int_0^1\frac{Li_3(1-x)}{1+x}dx\overset{1-x=>x}{\cong}\int_0^1\frac{Li_3(x)}{2-x}dx=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}Li_3(x)dx=$$
$$\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right)=\zeta(3)Li_1\left(\frac{1}{2}\right)-\zeta(2)Li_2\left(\frac{1}{2}\right)+\sum_{n=1}^{\infty}\frac{H_n}{2^n n^3}$$
$$\overset{\left\{\int_0^{\frac{1}{2}}x^{n-1}ln^2(x)dx=\frac{ln^2(2)}{n2^n}+\frac{2\ln(2)}{n^2 2^n}+\frac{2}{n^3 2^n}\right\}}{\cong}\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{\pi^2}{12}ln^2(2)-\frac{1}{6}ln^4(2)$$
$$-\frac{1}{2}ln^2(2)\sum_{n=1}^{\infty}\frac{H_n}{n2^n}-$$
$$\ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}-\frac{1}{6}\underbrace{\int_0^{\frac{1}{2}}\frac{ln^3(x)}{1-x}dx}_{\left\{\int_0^{\frac{1}{2}}\frac{ln^n(x)}{1-x}dx=\sum_{n=0}^{a}(-1)^n n!\binom{a}{n}((-\ln(2))^{a-n}Li_{n+1}(\frac{1}{2})\right\}}-\frac{1}{2}\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-x)ln^2(x)}{1-x}dx}_{-\frac{1}{4}ln^4(2)-\frac{1}{4}\zeta(4)}=$$
$$\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{\pi^2}{12}ln^2(2)+Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{720}-\frac{1}{8}\ln(2)\zeta(3)+\frac{1}{24}ln^4(2)=$$
$$\frac{7}{8}\zeta(3)\ln(2)-\frac{\pi^4}{80}+\frac{\pi^2}{12}ln^2(2)+Li_4\left(\frac{1}{2}\right)+\frac{ln^4(2)}{24}$$

$$J_4=\int_0^1\frac{\ln(1-x)Li_2(1-x)}{1+x}dx\overset{1-x\to x}{\cong}\int_0^1\frac{\ln(x)Li_2(x)}{2-x}dx$$
$$=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}\ln(x)Li_2(x)dx=$$
$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial}{\partial n}\int_0^1 x^{n-1}Li_2(x)dx\overset{\left\{\int_0^1 x^{n-1}Li_2(x)dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right\}}{\cong}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial}{\partial n}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right)=$$
$$\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{2H_n}{n^3}+\frac{H^{(2)}{}_n}{n^2}-\frac{2\zeta(2)}{n^2}\right)=3Li_4\left(\frac{1}{2}\right)-\frac{7\pi^4}{288}+\frac{ln^4(2)}{8}+\frac{\pi^2}{8}ln^2(2)$$

$$J_5=\frac{1}{2}\int_0^1\frac{\ln(x)ln^2(1-x)}{1+x}dx\overset{1-x\to x}{\cong}\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{2-x}dx$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\frac{\partial^2}{\partial n^2}x^{n-1}\ln(1-x)\,dx$$
$$\overset{\left\{\int_0^1 x^{n-1}\ln(1-x)dx=-\frac{H_n}{n}\right\}}{\cong}\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial^2}{\partial n^2}\left(-\frac{H_n}{n}\right)$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{2\zeta(3)}{n}+\frac{2\zeta(2)}{n^2}-\frac{2H_n}{n^3}-\frac{2H^{(2)}_n}{n^2}-\frac{2H^{(3)}_n}{n}\right)$$
$$2J_5=2\ln(2)\zeta(3)+2\zeta(2)Li_2\left(\frac{1}{2}\right)-2\sum_{n=1}^{\infty}\frac{H_n}{2^n n^3}-2\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{2^n n^2}-2\sum_{n=1}^{\infty}\frac{H^{(3)}_n}{2^n n}$$

$$=2\ln(2)\zeta(3)+\frac{\pi^4}{36}-\frac{\pi^2}{6}ln^2(2)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}-6Li_4\left(\frac{1}{2}\right)-Li_2^2\left(\frac{1}{2}\right)+4\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}-\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx$$

$$=2\ln(2)\zeta(3)+\frac{\pi^4}{36}-\frac{\pi^2}{6}ln^2(2)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}-6Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{144}-\frac{ln^4(2)}{4}+\frac{\pi^2}{12}ln^2(2)-\underbrace{\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx}_{T}$$

$$T=\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx=\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(1-x^2)}{x}dx}_{1-x^2\to x}-\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(\frac{1-x}{1+x})}{x}dx}_{x\to\frac{1-x}{1+x}}-\frac{1}{3}\underbrace{\int_0^1\frac{ln^3(1+x)}{x}dx}$$

$$=2Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{144}+\frac{7}{4}\ln(2)\zeta(3)-\frac{\pi^2}{12}ln^2(2)+\frac{1}{12}ln^4(2)$$

$$2J_5=-6Li_4\left(\frac{1}{2}\right)+\frac{11\pi^4}{360}-\frac{1}{4}ln^4(2)\to J_5=-3Li_4\left(\frac{1}{2}\right)+\frac{11\pi^4}{720}-\frac{1}{8}ln^4(2)\ J_6=\zeta(3)\int_0^1\frac{1}{1+X}dx=\zeta(3)\ln(2)$$

$$\Omega_1=\ln(2)J_1-J_2+J_3-J_4-J_5-\zeta(3)J_6=3Li_4\left(\frac{1}{2}\right)-\frac{29\pi^4}{1440}-\frac{\pi^2}{8}ln^2(2)+\frac{1}{8}ln^4(2)$$

$$\Omega_2=\sum_{n=1}^{\infty}\frac{(-1)^n\overrightarrow{H_n}}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\overleftrightarrow{H_n}\int_0^1x^{n-1}ln^2(x)dx=\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{x(1+x)}dx$$

$$=\frac{1}{2}\Big(\underbrace{\int_0^1\frac{\ln(1-x)ln^2(x)}{x}dx}_{IBP}-\int_0^1\frac{\ln(1-x)ln^2(x)}{1+x}dx\Big)=\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(x)}{1-x}dx}_{\left\{\int_0^1\frac{ln^a(x)}{1-x}dx=(-1)^a a!\zeta(a+1)\right\}}-\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1-x)ln^2(x)}{1+x}}_{-4Li_4(\frac{1}{2})+\zeta(4)+ln^2(2)\zeta(2)-\frac{1}{6}ln^4(2)}=2Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{60}-\frac{\pi^2}{12}ln^2(2)+\frac{1}{12}ln^4(2)$$

$$\Omega_3=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H_n}}{n^2}=-\sum_{n=1}^{\infty}(-1)^n\overline{H_n}\int_0^1x^{n-1}\ln(x)\,dx=-\int_0^1\frac{\ln(x)\ln(1-x)}{x(1+x)}dx$$

$$=\underbrace{\int_0^1\frac{\ln(x)\ln(1-x)}{1+x}dx}_{\frac{13}{8}\zeta(3)-\frac{3}{2}\ln(2)\zeta(2)}-\underbrace{\int_0^1\frac{\ln(x)\ln(1-x)}{x}dx}_{\zeta(3)}=\frac{13}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)-\zeta(3)=\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)$$

$$\Omega=\Omega_1-\Omega_2-{}_\gamma\Omega_3=3Li_4\left(\frac{1}{2}\right)-\frac{29\pi^4}{1440}-\frac{\pi^2}{8}ln^2(2)+\frac{1}{8}ln^4(2)-2Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{60}+\frac{\pi^2}{12}ln^2(2)-\frac{1}{12}ln^4(2)-{}_\gamma\Big(\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)\Big)$$

Answer:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n\psi^{(0)}(n)}{n^2}=Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{2880}+\frac{1}{24}ln^4(2)-\frac{\pi^2ln^2(2)}{24}-{}_\gamma\Big(\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)\Big)$$

Note section :

Harmonic numbers: $\sum_{k=1}^{n}\frac{1}{k}=H_n\ ;\ \sum_{k=1}^{n}\frac{1}{k^m}=H_n^{(m)}$

$$\frac{\partial}{\partial n}H_n = \zeta(2) - H_n^{(2)};\ \frac{\partial^2}{\partial n^2}H_n = 2H_n^{(3)} - 2\zeta(3);\ \sum_{n=1}^{\infty}(\pm x)^n H_n = -\frac{\ln(1 \pm x)}{1 \pm x}$$

$H_n = \psi^{(0)}(n+1) + \gamma$

Skew harmonic numbers:

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} = \ln(2) - \int_0^1 \frac{(-x)^n}{1+x} = \overline{H_n}$$

$$\sum_{n=1}^{\infty}(-1)^n \overline{H_n} x^n = \frac{\ln(1-x)}{1+x}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n2^n} = \frac{\zeta(2)}{2};\ \sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n} = \zeta(3) - \frac{1}{2}\ln(2)\zeta(2)$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2 2^n} + 2\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n} = 3Li_4\left(\frac{1}{2}\right) + \frac{1}{2}Li_2^2\left(\frac{1}{2}\right)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n} = Li_4\left(\frac{1}{2}\right) + \ln(2)\,Li_3\left(\frac{1}{2}\right) - \frac{1}{2}Li_2^2\left(\frac{1}{2}\right)$$

Polylogarithm function:

$$Li_a(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^a};\ Li_a(1) = \zeta(a)\ Li_2\left(\frac{1}{2}\right) = \frac{\pi^{12}}{12} - \frac{ln^2(2)}{2}$$

$$\frac{\partial}{\partial z}Li_a(z) = \frac{Li_{a-1}(z)}{z};\ Li_1(z) = -\ln(1-z);\ \int_0^1 x^{n-1}Li_2(x)dx = \frac{\zeta(2)}{n} - \frac{H_n}{n^2}$$

$$\int_0^1 x^{n-1}Li_3(x)dx = \frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}$$

$$\int_0^{\frac{1}{2}} x^{n-1}ln^2(x)dx = \frac{ln^2(2)}{2^n n} + \frac{2\ln(2)}{2^n n^2} + \frac{2}{2^n n^3}$$

2365. If $\alpha > 2$, *then* :

$$2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } x \geq \alpha > 2 \text{ and now, } \forall\, x \in (2,\infty), \frac{1-\cos x}{x^3} \overset{?}{<} \frac{2}{5x}$$

$$\Leftrightarrow 2x^2 + 5\cos x - 5 \underbrace{\overset{?}{>}}_{(*)} 0$$

$$\text{Let } F(x) = 2x^2 + 5\cos x - 5\ \forall\, x \in [2,\infty\rangle \text{ and then : } F'(x) = 4x - 5\sin x$$

$$\overset{\sin x \leq 1}{\geq} 4x - 5 \overset{x \geq 2}{\geq} 3 > 0 \Rightarrow F(x) \text{ is } \uparrow \text{ on } [2,\infty\rangle \Rightarrow F(x) \geq F(2) = 8 + 5\cos 2 - 5$$

$\approx 0.919 > 0 \therefore \forall x \in (2,\infty), 2x^2 + 5\cos x - 5 > 0 \Rightarrow (*)$ is true

$$\therefore \forall x \in (2,\infty), \frac{1-\cos x}{x^3} < \frac{2}{5x} \Rightarrow 2\int_{\alpha}^{2\alpha} \frac{1-\cos x}{x^3}\,dx < 2\int_{\alpha}^{2\alpha} \frac{2}{5x}\,dx = \frac{4}{5}(\ln 2\alpha - \ln\alpha)$$

$$= \frac{4}{5}.\ln 2 \Rightarrow e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < e^{\left(\frac{4}{5}\ln 2\right)} = \sqrt[5]{16} \therefore 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx}$$

$$< 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + \sqrt[5]{16} \overset{?}{<} 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

$$\Leftrightarrow \boxed{(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right) - \alpha.\cos\left(\frac{\pi}{\alpha}\right) \underset{(\bullet)}{\overset{?}{>}} \frac{\sqrt[5]{16}}{2}}$$

Let $f(t) = t.\cos\frac{\pi}{t}$ and then : $f'(t) = \frac{\pi}{t}.\sin\frac{\pi}{t} + \cos\frac{\pi}{t} \therefore$ LHS of $(\bullet)$

$$= f(\alpha+1) - f(\alpha) \overset{MVT}{=} ((\alpha+1)-\alpha)\frac{\pi}{\xi}.\sin\frac{\pi}{\xi} + \cos\frac{\pi}{\xi}\ (2 < \alpha < \xi < \alpha+1)$$

$\therefore$ LHS of $(\bullet) = \frac{\pi}{\xi}.\sin\frac{\pi}{\xi} + \cos\frac{\pi}{\xi} \to (1)$ and $\because \xi > 2 \therefore 0 < \frac{\pi}{\xi} < \frac{\pi}{2}$ and let

$P(\theta) = \theta.\sin\theta + \cos\theta\ \forall\theta \in \left[0,\frac{\pi}{2}\right) \therefore P'(\theta) = \theta.\cos\theta \ge 0 \Rightarrow P(x)$ is $\uparrow$ on $\left[0,\frac{\pi}{2}\right)$

$\Rightarrow P(x) \ge P(0) = 1 \therefore \forall\theta \in \left(0,\frac{\pi}{2}\right), \theta.\sin\theta + \cos\theta > 1 \therefore$ via (1), LHS of $(\bullet) > 1$

$$> \frac{\sqrt[5]{16}}{2} \Rightarrow (\bullet) \text{ is true} \therefore 2\alpha.\cos\left(\frac{\pi}{\alpha}\right) + e^{2\int_{\alpha}^{2\alpha}\frac{1-\cos x}{x^3}dx} < 2(\alpha+1).\cos\left(\frac{\pi}{\alpha+1}\right)$$

$\forall\alpha > 2$ (QED)

2366.

$$If \int_0^1 \frac{arctan^2(x)\log\left(\frac{x}{1-x^2}\right)}{1+x^2}dx = \frac{7\pi\zeta(a)}{128} - \frac{\pi\zeta(b)\ln(2)}{32}$$

then prove that :

$$\int_0^\infty \frac{(Li_{a+b}(-x) - Li_{a-b}(-x))}{1+x^2}dx = G + \frac{\pi}{4}\ln(2) - \frac{15\pi}{1024}\zeta(5) - 5\beta(6)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\int_0^1 \frac{arctan^2(x)\log\left(\frac{x}{1-x^2}\right)}{1+x^2}dx \overset{\{arctan(x)\to x\}}{=} \int_0^{\frac{\pi}{4}} x^2\ln\left(\frac{2\tan(x)}{2(1-tan^2(x))}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} x^2\ln\left(\frac{\tan(2x)}{2}\right)dx =$$

$$\int_0^{\frac{\pi}{4}} x^2\ln(\tan(2x))-\ln(2))dx=\int_0^{\frac{\pi}{4}} x^2\ln(\tan(2x))-\ln(2))dx-\ln(2)\int_0^{\frac{\pi}{4}} x^2dx \overset{2x\to x}{\cong}$$

$$\frac{1}{8}\int_0^{\frac{\pi}{2}} x^2\ln(\tan(x))\,dx-\ln(2)\frac{\pi}{4}\frac{x^3}{3}\Big|_0=\frac{1}{8}J-\frac{\pi^3\ln(2)}{192}$$

$$J\overset{Fourier\ series}{\cong}-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\underbrace{\int_0^{\frac{\pi}{2}} x^2\cos\big(2x(2n+1)\big)\,dx}_{IBP}=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\Big(\frac{\pi^2\sin(2\pi n)}{8(2n+1)^2}$$

$$-\frac{\sin(2\pi n)}{4(2n+1)^3}+\frac{\pi\cos(2\pi n)}{4(2n+1)^2}\Big)=\underbrace{\frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{\sin(2\pi n)}{(2n+1)^3}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{\sin(2\pi n)}{(2n+1)^4}}_{\{\sin(2\pi n)=0,n\in Z\}}+\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\cos(2\pi n)}{(2n+1)^3}$$

$$=\frac{\pi}{4}\Big(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n^3}+\sum_{n=0}^{\infty}\frac{1}{n^3}\Big)=\frac{7\pi\zeta(3)}{16}$$

$$\Omega=\frac{1}{8}J-\frac{\pi^3\ln(2)}{192}=\frac{7\pi\zeta(3)}{128}-\frac{\pi\zeta(2)\ln(2)}{32}\ ;\ \Omega=\frac{7\pi\zeta(a)}{128}-\frac{\pi\zeta(b)\ln(2)}{32}\ \{a=3,b=2\}$$

$$I=\int_0^{\infty}\frac{(Li_{a+b}(-x)-Li_{a-b}(-x))}{1+x^2}dx=\int_0^{\infty}\frac{Li_5(-x)}{1+x^2}dx-\int_0^{\infty}\frac{Li_1(-x)}{1+x^2}dx=I_1-I_2$$

$$I_1=\int_0^{\infty}\frac{Li_5(-x)}{1+x^2}dx$$

$$=\int_0^{\infty}\frac{1}{1+x^2}\Big(-\frac{1}{24}\int_0^1\frac{x\ln^4(y)}{1+xy}dy\Big)dx$$

$$=-\frac{1}{24}\int_0^1 \ln^4(y)\Big(\int_0^{\infty}\frac{1}{(1+xy)(1+x^2)}dx\Big)dy=$$

$$-\frac{1}{24}\int_0^1 \ln^4(y)\left(\frac{\pi}{2}\frac{y}{1+y^2}-\frac{\ln(y)}{1+y^2}\right)dy=-\frac{\pi}{48}\int_0^1\frac{y\ln^4(y)}{1+y^2}dy+\frac{1}{24}\int_0^1\frac{\ln^5(y)}{1+y^2}dy=-\frac{\pi}{48}M+\frac{1}{24}N$$

$$M=\int_0^1\frac{y\ln^4(y)}{1+y^2}dy$$

$$=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{2n+1}\ln^4(y)dy\overset{IBP}{\cong}\frac{24}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^5}=\frac{24}{32}\eta(5)=\frac{45}{64}\zeta(5)$$

$$N=\int_0^1\frac{\ln^5(y)}{1+y^2}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{2n}\ln^5(y)dy\overset{IBP}{\cong}-120\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}=-120\beta(6)$$

$$I_1=-\frac{\pi}{48}M+\frac{1}{24}N=-\frac{15\pi\zeta(5)}{1024}-5\beta(6)$$

Dirichlet beta function : $\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}$

$$I_2=\int_0^{\infty}\frac{Li_1(-x)}{1+x^2}dx=-\int_0^{\infty}\frac{\ln(1+x)}{1+x^2}dx=-\int_1^{\infty}\frac{\ln(1+x)}{1+x^2}dx-\int_0^1\frac{\ln(1+x)}{1+x^2}dx=$$

$$\int_0^1\frac{\ln(x)}{1+x^2}dx-2\int_0^1\frac{\ln(1+x)}{1+x^2}dx=K-2P$$

$$K=\int_0^1\frac{\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G$$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx \overset{\{\arctan(x)\to x\}}{\triangleq} \int_0^{\frac{\pi}{4}} \ln(1+\tan(x))\,dx = \underbrace{\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2}cos\left(\frac{\pi}{4}-x\right)\right)dx}_{\frac{\pi}{4}-x\to x} -$$

$$\int_0^{\frac{\pi}{4}} \ln(\cos(x))\,dx = \frac{\ln(2)}{2}\int_0^{\frac{\pi}{4}} dx + \int_0^{\frac{\pi}{4}} \ln(\cos(x))\,dx = \frac{\pi\ln(2)}{8}$$

$$I_2 = K - 2P = -G - \frac{\pi}{4}\ln(2)$$

Answer:

$$\int_0^\infty \frac{Li_5(-x)}{1+x^2}dx - \int_0^\infty \frac{Li_1(-x)}{1+x^2}dx = I_1 - I_2 = G + \frac{\pi}{4}\ln(2) - \frac{15\pi}{1024}\zeta(5) - 5\beta(6)$$

Solution 2 by Pham Duc Nam-Vietnam

$$I = \int_0^1 \frac{arctan^2(x)\log(\frac{x}{1-x^2})}{1+x^2}dx = \int_0^1 \frac{arctan^2(x)\ln(\frac{2x}{1-x^2}) - arctan^2(x)\ln(2)}{1+x^2}dx =$$

$$\frac{1}{4}\int_0^1 \frac{arctan^2\left(\frac{2x}{1-x^2}\right)\ln(\frac{2x}{1-x^2})}{1+x^2}dx - \frac{1}{3}\ln(2)\,arctan^2(x)\Big|_0^1 = I_1 - \frac{\pi^3 ln(2)}{192}$$

$$I_1 = \frac{1}{4}\int_0^1 \frac{arctan^2\left(\frac{2x}{1-x^2}\right)\ln\left(\frac{2x}{1-x^2}\right)}{1+x^2}dx \;\; let: x = \frac{1-t}{1+t} \to dx = -\frac{2}{(1+t)^2} \to$$

$$I_1 = -\frac{1}{4}\int_0^1 \frac{arctan^2\left(\frac{1-x^2}{2x}\right)\ln\left(\frac{2x}{1-x^2}\right)}{1+x^2}dx \to$$

$$2I_1 = \frac{1}{4}\int_0^1 \frac{\ln\left(\frac{2x}{1-x^2}\right)\left(arctan^2\left(\frac{2x}{1-x^2}\right) - arctan^2\left(\frac{1-x^2}{2x}\right)\right)}{1+x^2}dx =$$

$$\frac{\pi}{8}\int_0^1 \frac{\ln(\frac{2x}{1-x^2})(2\arctan\left(\frac{2x}{1-x^2}\right) - \frac{\pi}{2})}{1+x^2}dx \overset{x\to\tan(x)}{\triangleq} \frac{\pi}{8}\int_0^1 \ln(\tan(2x))\left(4x - \frac{\pi}{2}\right)dx =$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{4}} x\ln(\tan(2x))dx - \frac{\pi^2}{16}\int_0^{\frac{\pi}{4}} \ln(\tan(2x))\,dx \overset{x\to 2x}{\triangleq} \frac{\pi}{8}\int_0^{\frac{\pi}{2}} x\ln(\tan(x))dx$$

$$-\frac{\pi^2}{32}\int_0^1 x\ln(\tan(x))dx =$$

$$\frac{\pi}{8}\cdot\frac{7}{8}\zeta(3) - \frac{\pi^2}{32}.0 = \frac{7\pi}{64}\zeta(3) \quad I_1 = \frac{7\pi}{128}\zeta(3) \;\; ----->>>>$$

$$I = \frac{7\pi}{128}\zeta(3) - \frac{\pi^3\ln(2)}{192} = \frac{7\pi}{128}\zeta(3) - \frac{\pi\zeta(2)\ln(2)}{32} \;\; => a = 3, b = 2$$

$$J = \int_0^\infty \frac{Li_5(-x) - Li_1(-x)}{1+x^2}dx = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx + \int_0^\infty \frac{\ln(1+x)}{1+x^2}dx = J_1 + J_2$$

$$J_1 = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx = -\frac{1}{24}\int_0^1 ln^4(t)dt\int_0^\infty \frac{x}{(1+x^2)(1+xt)}dx =$$

$$-\frac{1}{24}\int_0^1 \frac{ln^4(t)}{1+t^2}dt\int_0^\infty \left(\frac{x+t}{1+x^2} - \frac{t}{1+xt}\right)dx = -\frac{1}{24}\int_0^1 \frac{ln^4(t)}{1+t^2}dt(tarctan(x) +$$

$$\frac{1}{2}\ln(x^2+1)-\ln(1+xt)\Big)_0^\infty=-\frac{1}{24}\int_0^1\frac{ln^4(t)}{1+t^2}\left(\frac{\pi}{2}t-\ln(t)\right)dt=-\frac{\pi}{1536}\int_0^1\frac{ln^4(t^2)}{1+t^2}d(t^2)$$

$$+\frac{1}{24}\int_0^1\frac{ln^5(t)}{1+t^2}dt=-\frac{\pi}{1536}\int_0^1\frac{ln^4(t)}{1+t}+\frac{1}{24}\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}ln^5(t)dt$$

$$apply:\int_0^1\frac{ln^n(t)}{1+t}dt=\left(-\frac{1}{2}\right)^n(2^n-1)\zeta(n+1)\Gamma(n+1)n=4=>$$

$$\int_0^1\frac{ln^4(t)}{1+t}dt=\frac{1}{16}.15.\zeta(5).\Gamma(5)=\frac{45}{2}\zeta(5)\quad=>$$

$$J_1=-\frac{\pi}{1536}.\frac{45}{2}\zeta(5)-\frac{120}{24}.\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^6}=-\frac{15\pi}{1024}\zeta(5)-5\beta(6)$$

$$J=G+\frac{\pi}{4}\ln(2)-\frac{15\pi}{1024}\zeta(5)-5\beta(6)$$

2367. Find:

$$\Omega=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{2^{n+k}(n+k)^3}=-\frac{1}{8}Li_3(\frac{1}{4})$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{2^{n+k}(n+k)^3}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 x^{n-1}log^2(x)\underbrace{\sum_{k=1}^{\infty}\frac{x^k}{2^k}}_{GS}dx$$

$$\frac{1}{2}\int_0^1\frac{log^2(x)}{2-x}\sum_{n=1}^{\infty}\frac{(-1)^nx^n}{2^n}dx=-\frac{1}{2}\int_0^1\frac{xlog^2(x)}{(2-x)(2+x)}dx=$$

$$-\frac{1}{2}\int_0^1\frac{xlog^2(x)}{4-x^2}dx=-\frac{1}{8}\int_0^1\frac{xlog^2(x)}{1-(\frac{x}{2})^2}dx=-\frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{4^n}\int_0^1 x^{2n+1}log^2(x)dx=$$

$$-\frac{2}{16.4}\sum_{n=0}^{\infty}\frac{1}{4^n(n+1)^3}=-\frac{1}{32}\sum_{n=1}^{\infty}\frac{4}{4^nn^3}=-\frac{1}{8}Li_3(\frac{1}{4})$$

2368. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{(2n+1)log\left(\frac{n^n}{n!}\right)}{n^3sin\left(\frac{\pi}{n}\right)}\right)$$

Proposed by Khaled Abd Imouti-Syria

Solution by Daniel Sitaru-Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{(2n+1)log\left(\frac{n^n}{n!}\right)}{n^3 sin\left(\frac{\pi}{n}\right)}\right)=\lim_{n\to\infty}\frac{1}{\pi}\cdot\frac{2n+1}{n}\cdot\frac{\frac{\pi}{n}}{sin\left(\frac{\pi}{n}\right)}\cdot\frac{log\left(\frac{n^n}{n!}\right)}{n}=$$

$$=\frac{1}{\pi}\cdot 2\cdot 1\cdot\lim_{n\to\infty}\frac{log\left(\frac{n^n}{n!}\right)}{n}\overset{TCS}{=}\frac{2}{\pi}\lim_{n\to\infty}\frac{log\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)-log\left(\frac{n^n}{n!}\right)}{n+1-n}=$$

$$=\frac{2}{\pi}\lim_{n\to\infty}log\left(\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right)=$$

$$=\frac{2}{\pi}\lim_{n\to\infty}log\left(\frac{(n+1)^n}{n^n}\right)=\frac{2}{\pi}\lim_{n\to\infty}log\left(1+\frac{1}{n}\right)^n=\frac{2}{\pi}loge=\frac{2}{\pi}$$

2369. Prove that:

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\log(x+y)}{\sqrt{x+y}}dx\,dy\,dz=8\sqrt{2}\log(2)-64\left(\frac{\sqrt{2}-1}{3}\right)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\left\{\frac{\log(x+y)}{\sqrt{x+y}}+\frac{\log(x+z)}{\sqrt{x+z}}+\frac{\log(z+y)}{\sqrt{z+y}}\right\}dx\,dy\,dz=$$

$$=\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}dx\,dy\,dz+\int_0^1\int_0^1\int_0^1\frac{\log(x+z)}{\sqrt{x+z}}dx\,dy\,dz=$$

$$+\int_0^1\int_0^1\int_0^1\frac{\log(z+y)}{\sqrt{z+y}}dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}dx\,dy\,dz=$$

$$=3\int_0^1\int_0^1\int_0^1\frac{\log(x+y)}{\sqrt{x+y}}dy\,dx$$

Let $\sqrt{x+y}=m$

$$I=12\int_0^1\int_{\sqrt{x}}^{\sqrt{x+1}}\log(m)\,dm\,dx=$$

$$=12\int_0^1\{\sqrt{x+1}\log(\sqrt{x+1})-\sqrt{x+1}-\sqrt{x}\log(\sqrt{x})+\sqrt{x}\}dx=$$

$$=12\left\{\frac{2\sqrt{2}}{3}\log(2)-\frac{16\sqrt{2}}{9}+\frac{16}{9}\right\}=8\sqrt{2}\log(2)-64\left(\frac{\sqrt{2}-1}{3}\right)$$

2370. Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{dxdydz}{\sqrt{x+2y+3z}}$$

Proposed by Ankush Kumar Parcha-India

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^1\int_0^1\int_0^1\frac{dxdydz}{\sqrt{x+2y+3z}}=\int_0^1\int_0^1(2\sqrt{x+2y+3z})\Big|_0^1dydz=\int_0^1(2\sqrt{x+2y+3z})-\sqrt{2y+3z})dy\Big|_0^1dz$$

$$\int_0^1 2\left(\frac{(2y+3z+1)^{\frac{3}{2}}}{\frac{3}{2}}\cdot\frac{1}{2}-\frac{(2y+3z)^{\frac{3}{2}}}{\frac{3}{2}}\cdot\frac{1}{2}\right)\Big|_0^1dz$$

$$=\int_0^1\left[\left(\frac{2}{3}(3+3z)^{\frac{3}{2}}-\frac{2}{3}(3z+2)^{\frac{3}{2}}-\left(\frac{2}{3}(3z+1)^{\frac{3}{2}}-\frac{2(3z)^{\frac{3}{2}}}{3}\right]dz$$

$$\frac{2}{3}\int_0^1\left[(3+3z)^{\frac{3}{2}}-(3z+2)^{\frac{3}{2}}-((3z+1)^{\frac{3}{2}}-(3z)^{\frac{3}{2}})\right]dz=\frac{2}{3}\left[\frac{(3+3z)^{\frac{5}{2}}}{\frac{5}{2}}\cdot\frac{1}{3}-\frac{(3z+2)^{\frac{5}{2}}}{\frac{5}{2}}\cdot\frac{1}{3}\right]_0^1=$$

$$\frac{4}{45}(36\sqrt{6}-25\sqrt{5}-16\sqrt{4}+9\sqrt{3}-9\sqrt{3}+4\sqrt{2}+1)=\frac{4}{45}(36\sqrt{6}-25\sqrt{5}+4\sqrt{2}-31)$$

2371. Find:

$$\Omega(x)=\int_1^e \frac{\sqrt{\log x}}{\sqrt{\log(1+e-x)}+\sqrt{\log x}}dx$$

Proposed by Khaled Abd Imouti-Syria

Solution by Daniel Sitaru-Romania

$$\Omega(x)=\int_1^e \frac{\sqrt{\log x}}{\sqrt{\log(1+e-x)}+\sqrt{\log x}}dx$$

$$y=1+e-x,\qquad dx=-dy$$

$$x=1\Rightarrow y=e$$

$$x=e\Rightarrow y=1$$

$$\Omega(x)=-\int_e^1 \frac{\sqrt{\log(1+e-y)}}{\sqrt{\log\big(1+e-(1+e-y)\big)}+\sqrt{\log(1+e-y)}}(-dy)=$$

$$=\int_1^e \frac{\sqrt{\log(1+e-y)}}{\sqrt{\log y}+\sqrt{\log(1+e-y)}}dy=\int_1^e \frac{\sqrt{\log(1+e-x)}}{\sqrt{\log x}+\sqrt{\log(1+e-x)}}dx$$

$$\Omega(x)=\int_1^e \frac{\sqrt{\log x}}{\sqrt{\log(1+e-x)}+\sqrt{\log x}}dx$$

$$\Omega(x)=\int_1^e \frac{\sqrt{\log(1+e-x)}}{\sqrt{\log x}+\sqrt{\log(1+e-x)}}dx$$

$$2\Omega(x)=\int_1^e \frac{\sqrt{\log x}}{\sqrt{\log(1+e-x)}+\sqrt{\log x}}dx+\int_1^e \frac{\sqrt{\log(1+e-x)}}{\sqrt{\log x}+\sqrt{\log(1+e-x)}}dx$$

$$2\Omega(x)=\int_1^e \frac{\sqrt{\log x}+\sqrt{\log(1+e-x)}}{\sqrt{\log(1+e-x)}+\sqrt{\log x}}dx$$

$$2\Omega(x)=\int_1^e dx$$

$$2\Omega(x)=e-1$$

$$\Omega(x)=\frac{e-1}{2}$$

2372. Find:

$$\Omega=\int_0^{\pi}\left(sin^4(x)+cos^2(x)+\frac{x^2}{`1+x^2}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^{\pi}\left(sin^4(x)+cos^2(x)+\frac{x^2}{`1+x^2}\right)dx=\Omega_1+\Omega_2+\Omega_3$$

$$\Omega_1=\int_0^{\pi}sin^4(x)dx\overset{\cos(x)\to t,\ \frac{dt}{dx}=-\sqrt{1-t^2}}{\cong}\int_{-1}^{1}\frac{(1-t^2)^2}{\sqrt{1-t^2}}dt=2\int_0^1(1-t^2)^{\frac{3}{2}}dt\overset{t^2\to t}{\cong}$$

$$\int_0^1 t^{-\frac{1}{2}}(1-t^2)^{\frac{3}{2}}dt=\beta\left(\frac{1}{2};\frac{5}{2}\right)=\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}=\frac{\Gamma(1+\frac{3}{2})\Gamma(1-\frac{3}{2})\sqrt{\pi}}{2\Gamma(-\frac{1}{2})}=-\frac{\frac{3\pi}{2}\csc\left(\frac{3\pi}{2}\right)}{4}=\frac{3\pi}{8}$$

$$\Omega_2=\int_0^{\pi}cos^2(x)dx=2\int_0^1 t^2\left(1-t^2\right)^{-\frac{1}{2}}dt=\int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\beta\left(\frac{3}{2};\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\Gamma\left(\frac{1}{2}\right)=\frac{\pi}{2}$$

$$\Omega_3=\int_0^{\pi}\frac{x^2}{1+x^2}dx=\int_0^{\pi}dx-\int_0^{\pi}\frac{1}{1+x^2}dx=\pi-\arctan(x)$$

$$\Omega_1+\Omega_2+\Omega_3=\frac{3\pi}{8}+\frac{\pi}{2}+\pi-\arctan(x)=\frac{15\pi}{8}-\arctan(x)$$

Solution 2 by Cosghun Mammedov-Azerbaijan

$$\Omega=\int_0^{\pi}\left(sin^4(x)+cos^2(x)+\frac{x^2}{`1+x^2}\right)dx=\int_0^{\pi}\left((sin^4(x)-sin^2(x)+1)+\frac{x^2}{`1+x^2}\right)dx=$$

$$\Omega=\int_0^{\pi}(sin^4(x)-sin^2(x)+1)dx+\int_0^{\pi}\frac{x^2}{`1+x^2}dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^{\pi}\left(sin^4(x)-sin^2(x)+\frac{1}{4}+\frac{3}{4}\right)dx=\int_0^{\pi}\left(\left(sin^2(x)-\frac{1}{2}\right)^2+\frac{3}{4}\right)dx=$$

$$\int_0^{\pi}\left(\left(\frac{1-\cos(2x)}{2}-\frac{1}{2}\right)^2+\frac{3}{4}\right)dx=\int_0^{\pi}\left(\left(-\frac{\cos(2x)}{2}\right)^2+\frac{3}{4}\right)dx=\int_0^{\pi}\left(\frac{cos^2(2x)}{4}+\frac{3}{4}\right)dx=$$

$$\int_0^{\pi}\left(\frac{\cos(4x)}{8}+\frac{7}{8}\right)dx=\left(\frac{1}{32}\sin(4x)+\frac{7}{8}x\right)\Big|_0^{\pi}=\frac{7}{8}\pi$$

$$\Omega_2=\int_0^{\pi}\frac{x^2}{1+x^2}dx=\int_0^{\pi}\left(1-\frac{1}{1+x^2}\right)dx=(x-\arctan(x))\Big|_0^{\pi}=\pi-\arctan(\pi)$$

$$\Omega=\Omega_1+\Omega_2=\frac{7}{8}\pi+\pi-\arctan(\pi)=\frac{15}{8}\pi-\arctan(\pi)$$

2373. Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx=\int_0^{\frac{\pi}{4}}\frac{x(1-\sin(2x))}{1+\cos(2x)}\overset{2x\to t}{\triangleq}\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{t(1-\sin(t))}{1+\cos(t)}dt=$$

$$\frac{1}{4}\left(\int_0^{\frac{\pi}{2}}\frac{t}{1+\cos(t)}dt-\int_0^{\frac{\pi}{2}}\frac{t\sin(t)}{1+\cos(t)}dt\right)$$

$$\Omega_1=\int_0^{\frac{\pi}{2}}\frac{t}{1+\cos(t)}dt\overset{IBP}{\triangleq}\left[t\tan\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\tan\left(\frac{t}{2}\right)dt=\frac{\pi}{2}+2\left[\log\left(\cos\left(\frac{x}{2}\right)\right)\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}-\ln(2)$$

$$\Omega_2=\int_0^{\frac{\pi}{2}}\frac{t\sin(t)}{1+\cos(t)}dt$$

$$=\int_0^{\frac{\pi}{2}}t\tan\left(\frac{t}{2}\right)dt\overset{IBP}{\triangleq}-\left[2t\ln\left(\cos\left(\frac{t}{2}\right)\right)\right]_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\ln\left(\cos\left(\frac{t}{2}\right)\right)dt=$$

$$\frac{\pi}{2}\ln(2)-2\ln(2)\int_0^{\frac{\pi}{2}}dt=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{2}}\cos(nt)\,dt\overset{IBP}{\triangleq}\frac{\pi}{2}\ln(2)-\pi\ln(2)-$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin(nt)}{n}\right]_0^{\frac{\pi}{2}}=-\frac{\pi}{2}\ln(2)-2\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\frac{\pi n}{2})}{n^2}=2G-\frac{\pi}{2}\ln(2)$$

$$\Omega=\frac{1}{4}(\Omega_1-\Omega_2)=\frac{1}{4}\left(\frac{\pi}{2}-\ln(2)-2G+\frac{\pi}{2}\ln(2)\right)=\frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{G}{2}+\frac{\pi}{8}\ln(2)$$

$$\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx=\frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{G}{2}+\frac{\pi}{8}\ln(2)$$

2374. Prove that:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dxdydz}{(1+x^2y^2)(1+x^2z^2)(1+y^2z^2)}=\frac{\pi^3}{4\sqrt{2}}$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Cosghun Memmedov-Azerbaijan

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2y^2)(1+x^2z^2)(1+y^2z^2)} \overset{x\to 1/x}{\cong} \int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(x^2+y^2)(x^2+z^2)(y^2+z^2)} =$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \left(\frac{z^2}{x^2+z^2}-\frac{y^2}{x^2+y^2}\right)\frac{dxdydz}{(1+z^2y^2)(z^2-y^2)}$$

$$= \int_0^\infty\int_0^\infty \frac{1}{(1+z^2y^2)(z^2-y^2)}\left(z\tan^{-1}\left(\frac{x}{z}\right)-y\tan^{-1}\left(\frac{x}{y}\right)\right)_0^\infty dydz$$

$$\frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{1}{(y+z)(1+z^2y^2)}dydz \overset{IBP}{\cong}$$

$$= \frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(yz)}{z(y+z)^2}dydz \overset{(By\ symmetry)}{\cong} \frac{\pi}{4}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(yz)}{zy(y+z)}dydz$$

$$\underset{(\frac{dt}{dy}=z)}{\overset{(yz=t)}{\cong}} \frac{\pi}{4}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(t)}{t(t+z^2)}dydz = \frac{\pi}{4}\int_0^\infty \frac{\tan^{-1}}{t}dt\int_0^\infty \frac{1}{z^2+t}dz = \frac{\pi^2}{8}\int_0^\infty \frac{\tan^{-1}(t)}{t^{\frac{3}{2}}}dt =$$

$$-\frac{\pi^2}{4}\int_0^\infty \tan^{-1}(t)d(t^{-\frac{1}{2}}) \overset{IBP}{\cong} \frac{\pi^2}{4}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t^2}dt = \frac{\pi^2}{2}\int_0^\infty \frac{1}{1+t^4}dt = \frac{\pi^3}{4\sqrt{2}}$$

Solution 2 by Pham Duc Nam-Vietnam

$$I = \int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(1+x^2y^2)(1+y^2z^2)(1+x^2z^2)}dxdydz = \frac{\pi^3}{4\sqrt{2}}?$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\int_{-\infty}^\infty \frac{1}{(1+x^2y^2)(1+x^2z^2)}dx$$

$$= \int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\left(\pi i.\,Res\left(\frac{1}{(1+x^2y^2)(1+x^2z^2)}, x=\frac{i}{y}, x=\frac{i}{z}\right)\right)$$

$$= \int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\left(\pi i.\left(\begin{array}{l}\lim\limits_{x\to\frac{i}{y}}\left(x-\frac{i}{y}\right)\frac{1}{(1+x^2y^2)(1+x^2z^2)}+ \\ +\lim\limits_{x\to\frac{i}{z}}\left(x-\frac{i}{z}\right)\frac{1}{(1+x^2y^2)(1+x^2z^2)}\end{array}\right)\right)$$

$$= \int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\left(\pi i\left(-\frac{i}{2(y+z)}\right)\right) = \frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{1}{(y+z)(1+y^2z^2)}dydz$$

$$t = yz \Rightarrow y = \frac{t}{z} \Rightarrow dy = \frac{1}{z}dt \Rightarrow I = \frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{1}{\left(\frac{t}{z}+z\right)(1+t^2)}\frac{1}{z}dt\,dz =$$

$$=\frac{\pi}{z}\int_0^{\infty}\int_0^{\infty}\frac{1}{(t+z^2)(1+t^2)}dtdz$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{1}{1+t^2}dt\int_0^{\infty}\frac{1}{t+z^2}dz=\frac{\pi}{2}\int_0^{\infty}\frac{1}{1+t^2}dt\left(\frac{\pi}{2}\frac{1}{\sqrt{t}}\right)=\frac{\pi^2}{4}\int_0^{\infty}\frac{1}{\sqrt{t}(1+t^2)}dt=$$

$$=\frac{\pi^2}{2}\int_0^{\infty}\frac{1}{2\sqrt{t}(1+t^2)}dt$$

$$=\frac{\pi^2}{2}\int_0^{\infty}\frac{1}{(1+t^2)}d(\sqrt{t})=\frac{\pi^2}{2}\int_0^{\infty}\frac{1}{1+t^4}dt=\frac{\pi^2}{4}\int_{-\infty}^{\infty}\frac{t^2}{1+t^4}dt=$$

$$=\frac{\pi^2}{4}\int_{-\infty}^{\infty}\frac{1}{\left(t-\frac{1}{t}\right)^2+2}dt\overset{Glasser's\ master\ theorem}{\rightarrow}\frac{\pi^2}{4}\int_{-\infty}^{\infty}\frac{1}{t^2+2}dt$$

$$=\frac{\pi^2}{4}\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}t\right)\Big|_{-\infty}^{\infty}=\frac{\pi^3}{4\sqrt{2}}$$

2375. Find a closed form:

$$\Omega=\int_0^1\int_0^1 ln\left(\frac{x\sqrt{x}}{y\sqrt{y}}+\frac{y\sqrt{y}}{x\sqrt{x}}\right)dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\ln\left(\frac{x\sqrt{x}}{y\sqrt{y}}+\frac{y\sqrt{y}}{x\sqrt{x}}\right)dxdy=$$

$$=\int_0^1\int_0^1\ln(x^3+y^3)dxdy-\frac{3}{2}\int_0^1\int_0^1\ln(xy)dxdy=M-\frac{3}{2}K$$

$$M=\int_0^1\int_0^1 ln(x^3+y^3)dxdy=\int_0^1\ln(1+y^3)dy-3\int_0^1\int_0^1\frac{x^3}{x^3+y^3}dxdy=P-3Q$$

$$P=\int_0^1\ln(1+y^3)dy=\ln 2-3\int_0^1\frac{y^3}{1+y^3}dy=\ln(2)-3+3\int_0^1\frac{1}{1+y^3}dy=$$

$$=\ln(2)-3+3\left(\frac{1}{3}\ln(2)+\frac{\sqrt{3}}{9}\pi=2\ln(2)+\frac{\pi}{\sqrt{3}}-3\right.$$

$$Q=\int_0^1\int_0^1\frac{x^3}{x^3+y^3}dxdy=\int_0^1\int_0^1\frac{y^3}{x^3+y^3}dx\,dy$$

$$2Q=\int_0^1\int_0^1 \frac{x^3+y^3}{x^3+Y^3}dx\,dy=1 \leftrightarrow Q=\frac{1}{2}$$

$$M=P-3Q=2\ln(2)+\frac{\pi}{\sqrt{3}}-3-\frac{3}{2}=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{9}{2}$$

$$K=\int_0^1\int_0^1 \ln(xy)\,dxdy=\int_0^1\int_0^1 \ln(x)\,dxdy+\int_0^1\int_0^1 \ln(y)\,dxdy=-2$$

$$\Omega=M-\frac{3}{2}K=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{9}{2}+\frac{3}{2}x2=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{3}{2}$$

2376. Find:

$$\Omega(x)=\frac{\sum_{k=1}^{\infty}\left(\frac{\sin k}{k}\right)x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}},\qquad x\in[-1,1]$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Pham Duc Nam-Vietnam

$$\Omega(x)=\frac{\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}},x\in[-1,1]$$

$$*\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k=\Im\sum_{k=1}^{\infty}\frac{e^{ik}}{k}x^k=\Im\sum_{k=1}^{\infty}\frac{(xe^i)^k}{k}=\Im(-\ln(1-xe^i))$$

$$=\Im(-\ln(1-x\cos(1)-ix\sin(1)))=\arctan\left(\frac{x\sin(1)}{1-x\cos(1)}\right)$$

$$*\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}=\prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2}=\prod_{k=1}^{\infty}\left(1-\left(\frac{1}{2k}\right)^2\right)$$

$$\frac{\sin(\pi z)}{\pi z}=\prod_{k=1}^{\infty}\left(1-\left(\frac{z}{k}\right)^2\right),\ \text{let: } z=\frac{1}{2}\Rightarrow\prod_{k=1}^{\infty}\left(1-\left(\frac{1}{2k}\right)^2\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}=\frac{2}{\pi}$$

$$\Rightarrow\Omega(x)=\frac{\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}}=\frac{\pi}{2}\arctan\left(\frac{x\sin(1)}{1-x\cos(1)}\right)$$

2377. Find:

$$\Omega(a,b,c)=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)(a^x+b^x+c^x)^{\frac{1}{x}}}{3^{\frac{1}{x}}\left((1+x)^{\frac{1}{x}}-e\right)},a,b,c>0$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution 1 by Pham Duc Nam-Vietnam

$$\Omega(a,b,c)=\lim_{x\to 0}\frac{x\left(x\sin\left(\frac{1}{x}\right)+1\right)}{(1+x)^{\frac{1}{x}}-e}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}},a,b,c>0$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{x\left(x\sin\left(\frac{1}{x}\right)+1\right)^{\frac{1}{x\sin\left(\frac{1}{x}\right)}x\sin\left(\frac{1}{x}\right)}}{e^{\frac{1}{x}\ln(1+x)-1}-1}\left(x\frac{\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}+\frac{3}{x}}{3}\right)^{\frac{1}{x}}=$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{xe^{x\sin\left(\frac{1}{x}\right)}}{\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{\frac{1}{x}\ln(1+x)-1}}\frac{1}{\frac{1}{x}\ln(1+x)-1}\left(x\frac{\ln(abc)+\frac{3}{x}}{3}\right)^{\frac{1}{x}}$$

$$=\frac{1}{e}\lim_{x\to 0}e^{x\sin\left(\frac{1}{x}\right)}\frac{x}{\frac{1}{x}\ln(1+x)-1}\left(1+\frac{1}{3}x\ln(abc)\right)^{\frac{1}{x}}=$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{x^2}{\ln(1+x)-x}\left(1+\frac{1}{3}x\ln(abc)\right)^{\frac{3}{x\ln(abc)}\frac{1}{3}\ln(abc)}$$

$$=\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}\frac{x^2}{\ln(1+x)-x}\left(\frac{0}{0}\right)\overset{L'H}{\to}-\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}\frac{2x}{\frac{x}{1+x}}=$$

$$=-2\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}(1+x)=-\frac{2\sqrt[3]{abc}}{e}$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)(a^x+b^x+c^x)^{\frac{1}{x}}}{3^{\frac{1}{x}}\left((1+x)^{\frac{1}{x}}-e\right)}=\underbrace{\left(\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}-1\right)}{(1+x)^{\frac{1}{x}}-e}\right)}_{\frac{0}{0}}\underbrace{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}}_{1^{\infty}}=$$

$$=\left(-\frac{2}{e}\right)\left(\sqrt[3]{abc}\right)=-\frac{2\sqrt[3]{abc}}{e}$$

where

$$\text{(i) }\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{(1+x)^{\frac{1}{x}}-e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{\frac{1}{x}\ln(1+x)}-e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{\frac{1}{x}\left(x-\frac{1}{2}x^2+O(x^3)\right)}-e}=$$

$$=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{1-\frac{1}{2}x+O(x^2)}=e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(e^{-\frac{1}{2}x+O(x^2)}-1\right)}$$

$$=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(\left(1-\frac{1}{2}x+O(x^2)-1\right)\right)}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(-\frac{1}{2}x+O(x^2)\right)}=$$

$$=\lim_{x\to 0}\frac{x\sin\frac{1}{x}+1}{e\left(-\frac{1}{2}+O(x)\right)}=\frac{0+1}{e\left(-\frac{1}{2}\right)}=-\frac{2}{e}$$

$$\text{(ii) }\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}=\exp\left\{\underbrace{\lim_{x\to 0}\frac{\ln(a^x+b^x+c^x)-\ln 3}{x}}_{\frac{0}{0}}\right\}=\exp\left\{\underbrace{\lim_{x\to 0}\frac{\frac{a^x\ln a+b^x\ln b+c^x\ln c}{a^x+b^x+c^x}}{1}}_{L'Hopital\ Rule}\right\}$$

$$=\exp\left\{\frac{\ln a+\ln b+\ln c}{3}\right\}=e^{\ln(abc)^{\frac{1}{3}}}=\sqrt[3]{abc}$$

2378. **Prove that:**

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=Li_2\left(-\frac{1}{2}\right)-Li_3\left(-\frac{1}{2}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 x^{n+k-1}log^2(x)dx=$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\int_0^1 x^{k-1}log^2(x)\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{2^n}}_{GS}dx$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\int_0^1\left(-\frac{\frac{x}{2}}{1+\frac{x}{2}}\right)x^{k-1}log^2(x)dx=$$

$$-\frac{1}{2}\int_0^1\frac{log^2(x)}{x+2}\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{2^k}dx=\frac{1}{2}\int_0^1\frac{xlog^2(x)}{(x+2)^2}dx;\ \ f(x)=\sum_{n=1}^{\infty}(-\frac{x}{2})^n=-\frac{x}{x+2}$$

$$*\left\{\frac{\partial}{\partial x}f(x)=\sum_{n=1}^{\infty}\frac{n(-1)^n x^{n-1}}{2^n}=-\frac{2}{(x+2)^2}\ \ \sum_{n=1}^{\infty}\frac{n(-1)^n x^n}{2^n}=-\frac{2x}{(x+2)^2}\right\}$$

$$-\frac{1}{4}\sum_{n=1}^{\infty}\frac{n(-1)^n}{2^n}\int_0^1 x^n log^2(x)dx\overset{IBP}{\cong}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{n(-1)^n}{2^n(n+1)^3}=$$

$$-\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^3}\right)=-\frac{1}{2}\left(-2Li_2\left(-\frac{1}{2}\right)-1+2Li_3\left(-\frac{1}{2}\right)+1\right)=$$

$$=Li_2\left(-\frac{1}{2}\right)-Li_3\left(-\frac{1}{2}\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{2^{n+k}(n+k)^3}=\sum_{m\in N}(-1)^n\frac{(m-1)}{2^m m^3}=$$

$$\sum_{m\in N}(-\frac{1}{2})^m\cdot\frac{1}{m^2}-\sum_{m\in N}(-\frac{1}{2})^m\cdot\frac{1}{m^3}=Li_2\left(-\frac{1}{2}\right)-Li_3(-\frac{1}{2})$$

$$Note:\begin{cases}\displaystyle\sum_{a,b\in N}f(a+b)=\sum_{n\in N}(n-1)f(n)\\ \displaystyle And,\ \sum_{a,b\in N}f(a+b)=\sum_{a\in N}\sum_{b\in N}f(a+b)\end{cases}$$

2379. Prove the below closed form

$$\int_0^1 \frac{\tan^{-1}(x)}{\sqrt{1+x}}dx = \frac{\pi}{\sqrt{2}} - \sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Rana Ranino-Algerie

$$\Omega = \int_0^1 \frac{\tan^{-1} x}{\sqrt{1+x}}dx$$

$$\Omega = \left[2\tan^{-1}(x)\sqrt{1+x}\right]_0^1 - 2\int_0^1 \frac{\sqrt{1+x}}{1+x^2}dx = \frac{\pi}{\sqrt{2}} - 2\int_0^1 \frac{\sqrt{1+x}}{1+x^2}dx$$

$$\Omega \overset{1+x=t^2}{\cong} \frac{\pi}{\sqrt{2}} - 4\int_1^{\sqrt{2}} \frac{t^2}{t^4-2t^2+2}dt =$$

$$= \frac{\pi}{\sqrt{2}} - 2\int_1^{\sqrt{2}} \frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt - 2\int_0^{\sqrt{2}} \frac{t^2-\sqrt{2}}{t^4-2t^2+2}dt$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 2\int_1^{\sqrt{2}} \frac{\left(1+\frac{\sqrt{2}}{t^2}\right)}{\left(t-\frac{\sqrt{2}}{t}\right)^2 - 2 + 2\sqrt{2}}dt - 2\int_1^{\sqrt{2}} \frac{\left(1-\frac{\sqrt{2}}{t^2}\right)}{\left(t+\frac{\sqrt{2}}{t}\right)^2 - 2 - 2\sqrt{2}}dt$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 2\int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{u^2+2(\sqrt{2}-1)}du - 2\int_{1+\sqrt{2}}^{1-\sqrt{2}} \frac{1}{v^2-2(2\sqrt{2}+1)}dv =$$

$$= \frac{\pi}{\sqrt{2}} - 4\int_0^{\sqrt{2}-1} \frac{1}{u^2+2(\sqrt{2}-1)}du$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 4\left[\frac{1}{\sqrt{2(\sqrt{2}-1)}}\tan^{-1}\left(\frac{u}{\sqrt{2(\sqrt{2}-1)}}\right)\right]_0^{\sqrt{2}-1} =$$

$$= \frac{\pi}{\sqrt{2}} - 2\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)$$

Using identity: $2\tan^{-1}(a) = \tan^{-1}\left(\frac{2a}{1-a^2}\right)$

$$\Omega=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{\frac{\sqrt{2}-1}{2}}}{1-\frac{\sqrt{2}-1}{2}}\right)=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{4\sqrt{\sqrt{2}-1}}{3\sqrt{2}-2}\right)$$

$$\Omega=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{(\sqrt{2}-1)(22+12\sqrt{2})}}{7}\right)=$$

$$=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

$$\int_0^1\frac{\tan^{-1}x}{\sqrt{1+x}}dx=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{\tan^{-1}x}{\sqrt{x+1}}dx\overset{IBP}{=}2\left(\tan^{-1}(x)\sqrt{x+1}\right)\Big|_0^1-2\int_0^1\frac{\sqrt{x+1}}{x^2+1}dx=\frac{\pi}{\sqrt{2}}-2A$$

$$A=\int_0^1\frac{\sqrt{x+1}}{x^2+1}dx\overset{\sqrt{x+1}=t}{=}2\int_0^{\sqrt{2}}\frac{t^2}{t^4-2t^2+2}dt\overset{t\to\frac{\sqrt{2}}{t}}{=}2\int_0^{\sqrt{2}}\frac{\sqrt{2}}{t^4-2t^2+2}dt=$$

$$=\int_0^{\sqrt{2}}\frac{t^2}{t^4-2t^2+2}dt=\int_0^{\sqrt{2}}\frac{\sqrt{2}}{t^4-2t^2+2}dt=|\to 2|=\int_0^{\sqrt{2}}\frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt$$

$$A=2|=\int_0^{\sqrt{2}}\frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt=\int_0^{\sqrt{2}}\frac{\left(1+\frac{\sqrt{2}}{t^2}\right)}{t^2+\left(\frac{\sqrt{2}}{t}\right)^2-2}dt=$$

$$=\int_0^{\sqrt{2}}\frac{d\left(t-\frac{\sqrt{2}}{t}\right)}{\left(t-\frac{\sqrt{2}}{t}\right)^2+\left(\sqrt{2\sqrt{2}-2}\right)^2}=\frac{1}{\sqrt{2\sqrt{2}-2}}\tan^{-1}\left(\frac{\left(t-\frac{\sqrt{2}}{t}\right)}{\sqrt{2\sqrt{2}-2}}\right)\Bigg|_1^{\sqrt{2}}=$$

$$=\frac{\sqrt{2+\sqrt{2}}}{2}\left(\tan^{-1}\frac{\sqrt{2}-1}{\sqrt{2\sqrt{2}-2}}\right)-\tan^{-1}\left(\frac{1-\sqrt{2}}{\sqrt{2\sqrt{2}-2}}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2}}}{2}\tan^{-1}\left(\frac{\frac{2\sqrt{2}-1}{\sqrt{2\sqrt{2}-2}}}{1-\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}-2}}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}\tan^{-1}\left(\frac{\sqrt{(2\sqrt{2}-2)(12-8\sqrt{2})}}{4\sqrt{2}-5}\right)=$$

$$\Omega=\frac{\pi}{\sqrt{2}}-2A=\frac{\pi}{\sqrt{2}}-\sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Notes: $\tan^{-1}(x)\pm\tan^{-1}(y)=\tan\left(\frac{x\pm y}{1\mp xy}\right)$

Answer:

$$\int_0^1\frac{\tan^{-1}x}{\sqrt{x+1}}dx=\frac{\pi}{\sqrt{2}}-\sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

2380. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n}\ln\binom{n}{k}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Pham Duc Nam-Vietnam

$$L=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n}\ln\left(C_n^k\right)=\lim_{n\to\infty}\frac{1}{n^2}\ln\left(\prod_{k=0}^{n}C_n^k\right)$$

$$\overset{*}{=}\lim_{n\to\infty}\frac{\ln\left(\prod_{k=0}^{n+1}C_{n+1}^k\right)-\ln\left(\prod_{k=0}^{n}C_n^k\right)}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{\ln\left(\prod_{k=0}^{n}\frac{n+1}{n+k-1}\right)}{2n+1}$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)}{2n+1}=\lim_{n\to\infty}\frac{(n+1)\ln(n+1)-\ln\left((n+1)!\right)}{2n+1}$$

$$\overset{*}{=}\lim_{n\to\infty}\frac{(n+2)\ln(n+2)-\ln\left((n+2)!\right)-(n+1)\ln(n+1)+\ln\left((n+1)!\right)}{2n+3-2n-1}$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(n\ln\left(\frac{n+2}{n+1}\right)+2\ln\left(\frac{n+2}{n+1}\right)\right)=\frac{1}{2}\lim_{n\to\infty}\left(\ln\left(1+\frac{1}{n+1}\right)^{n+1\frac{n}{n+1}}+2\ln\left(\frac{n+2}{n+1}\right)\right)$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\ln\left(e^{\frac{n}{n+1}}\right)+2\ln\left(1+\frac{1}{n+1}\right)\right)=\frac{1}{2}$$

$$\Rightarrow L = \frac{1}{2} \text{ by Stolz – Cesaro theorem}$$

2381. Find a closed form:

$$\int_0^\infty \frac{x^2 + \sin(\pi x)}{1 + \exp(\pi x)} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\int_0^\infty \frac{x^2 + \sin(\pi x)}{1 + e^{\pi x}} dx = \int_0^\infty \frac{x^2}{1 + e^{\pi x}} dx + \int_0^\infty \frac{\sin(\pi x)}{1 + e^{\pi x}} dx = \Omega_1 + \Omega_2$$

$$\Omega_1 = \int_0^\infty \frac{x^2}{1 + e^{\pi x}} dx \ \left\{ e^{\pi x} = \frac{1}{t} \ \pi x = -\ln(t) \ dx = -\frac{1}{\pi t} \right\}$$

$$\Omega_1 = -\frac{1}{\pi^3} \int_0^1 \frac{ln^2(t)}{1 + \frac{1}{t}} \frac{dt}{t} = -\frac{1}{\pi^3} \int_0^1 \frac{ln^2(t)}{1 + t} dt = -\frac{1}{\pi^3} \sum_{n=0}^\infty (-1)^n \int_0^1 t^n ln^2(t) dt$$

$$\stackrel{IBP}{=} -\frac{2}{\pi^3} \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^3} = \frac{3\zeta(3)}{2\pi^3}$$

$$\Omega_2 = \int_0^\infty \frac{\sin(\pi x)}{1 + e^{\pi x}} dx \stackrel{\{\pi x = t\}}{=} \frac{1}{\pi} \int_0^\infty \frac{\sin(t)}{1 + e^t} dt = \frac{1}{\pi} I \ ; \ note \ \{\sinh(it) = i\sin(t)\}$$

$$I(a) = \int_0^\infty \frac{\sinh(at)}{1 + e^t} dt \stackrel{e^t \to t}{=} \underbrace{\frac{1}{2} \int_1^\infty \frac{t^a}{t(t+1)} dt}_{t \to \frac{1}{t}} - \frac{1}{2} \int_1^\infty \frac{t^{-a}}{t(t+1)} dt =$$

$$\frac{1}{2} \int_0^1 \frac{t^{-a}}{1+t} dt - \frac{1}{2} \int_1^\infty t^{-a} \left(\frac{1}{t} - \frac{1}{1+t} \right) dt = \frac{1}{2} \left(\underbrace{\int_0^\infty \frac{t^{-a}}{1+t} dt}_{\{\pi \csc(\pi a) \ \Re\{a\} < 1\}} - \int_1^\infty t^{-a-1} dt \right) =$$

$$\frac{1}{2} \left(\pi \csc(\pi a) - \frac{1}{a} \right); I(i) = \frac{1}{2} \left(\pi \csc(\pi i) - \frac{1}{i} \right) = \frac{i}{2} (1 - \pi \operatorname{csch}(\pi))$$

$$\Omega_2 = \frac{1}{\pi} \left(\int_0^\infty \frac{\sin(t)}{1 + e^t} dt \right) = -\frac{i}{\pi} \left(\int_0^\infty \frac{\sinh(it)}{1 + e^t} dt \right) = \frac{1}{2\pi} (1 - \pi \operatorname{csch}(\pi))$$

$$\int_0^\infty \frac{x^2 + \sin(\pi x)}{1 + e^{\pi x}} dx = \int_0^\infty \frac{x^2}{1 + e^{\pi x}} dx + \int_0^\infty \frac{\sin(\pi x)}{1 + e^{\pi x}} dx =$$

$$\frac{3\zeta(3)}{2\pi^3} + \frac{1}{2\pi} - \frac{1}{2} \operatorname{csch}(\pi)$$

Note : $\zeta(3) \to Apery's\ constant$

Solution 2 by Ankush Kumar Parcha-India

$$\text{We have, } \underbrace{\int_0^\infty \frac{x^2}{1+\exp(\pi x)}dx}_{\Omega_1} + \underbrace{\int_0^\infty \frac{\sin(\pi x)}{1+\exp(\pi x)}dx}_{\Omega_2}$$

$$\Omega_1 = \int_0^\infty \frac{x^2}{1+\exp(\pi x)}dx \overset{\pi x\to x}{\cong} \frac{1}{\pi^3}\int_0^\infty \frac{x^2}{1+\exp(x)}dx \overset{\text{Note section}}{\cong} \frac{1}{\pi^3}\sum_{n\in N}(-1)^n \mathcal{L}_x\{x^2\}(n)$$

$$\underbrace{\overset{\text{Note Section}}{\cong}}_{(2)} \frac{2}{\pi^3}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3} \Rightarrow \frac{2}{\pi^3}\eta(3) \overset{\text{note } \eta(s)=(1-2^{1-s}).\zeta(s)}{\cong} \int_0^\infty \frac{x^2}{1+\exp(\pi x)}dx = \frac{3\zeta(3)}{2\pi^3}$$

$$\Omega_2 = \int_0^\infty \frac{\sin(\pi x)}{1+\exp(\pi x)}dx \overset{\pi x\to x}{\cong} \frac{1}{\pi}\int_0^\infty \frac{\sin(x)}{1+\exp(x)}dx \overset{\text{note section (1)}}{\Longrightarrow}$$

$$\frac{1}{\pi}\sum_{n\in N}(-1)^{n-1}\mathcal{L}_x\{\sin(x)\}(n) \underbrace{\overset{\text{Note Section}}{\cong}}_{(3)} \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+1} = \frac{1}{\pi}\Im\sum_{n\in N}\frac{(-1)^{n-1}}{n-i} \underbrace{\overset{\text{Note Section}}{\cong}}_{(4)}$$

$$\frac{1}{2\pi}\Im\left[\psi^{(0)}\left(1-\frac{i}{2}\right) - \psi^{(0)}\left(\frac{1}{2}-\frac{i}{2}\right)\right]$$

$$\text{Note : } \left(\Im\{\psi^{(0)}(z)\} = \frac{\psi^{(0)}(z) - \overline{\psi^{(0)}(z)}}{2i};\ \overline{\psi^{(0)}(z)} = \psi^{(0)}\overline{(z)}\right)$$

$$\Rightarrow \frac{-i}{4\pi}\left[\psi^{(0)}\left(1-\frac{i}{2}\right) - \psi^{(0)}\left(1+\frac{i}{2}\right) + \psi^{(0)}\left(\frac{1}{2}+\frac{i}{2}\right) - \psi^{(0)}\left(\frac{1}{2}-\frac{i}{2}\right)\right] \overset{\text{Note section (5,6,7)}}{\Longrightarrow}$$

$$\frac{-i}{4\pi}\left[2i + \pi\cot\left(\frac{i\pi}{2}\right) + \pi\tan\left(\frac{i\pi}{2}\right)\right] \Rightarrow \frac{1}{2\pi} + \frac{1}{4}\left[\tanh\left(\frac{\pi}{2}\right) - \coth\left(\frac{\pi}{2}\right)\right] \to$$

$$\Omega_2 = \int_0^\infty \frac{\sin(\pi x)}{1+\exp(\pi x)}dx = \frac{1}{2\pi} - \frac{\operatorname{csch}(\pi)}{2}$$

$$\text{Put the values of } \Omega_1 \text{ and } \Omega_2 \text{ in equation} - (1).\ \text{We get,}$$

$$\Omega_1 + \Omega_2 = \int_0^\infty \frac{x^2+\sin(\pi x)}{1+\exp(\pi x)}dx = \frac{1}{2\pi} - \frac{\operatorname{csch}(\pi)}{2} + \frac{3\zeta(3)}{2\pi^3}$$

Note Section :

$$1.\ \text{Maz Summation Identity : } \sum_{n\in N}(-1)^{n-1}\mathcal{L}_t\{f(t)\}(n) = \int_0^\infty \frac{f(t)}{1+e^t}$$

$$2.\ \mathcal{L}_t\{t^a\}(s) = \frac{\Gamma(a+1)}{s^{a+1}},\ \Re(a) > -1$$

$$3\ \mathcal{L}_t(\sin(\omega t))(s) = \frac{\omega}{s^2+\omega^2}\quad s > |\Im(\omega)|$$

$$4.\ \sum_{n\in N}\frac{(-1)^{n-1}}{n-a} = \frac{1}{2}\left[\psi^{(0)}\left(1-\frac{a}{2}\right) - \psi^{(0)}\left(\frac{1}{2}-\frac{a}{2}\right)\right]$$

$$5.\ \psi^{(0)}\left(\frac{1}{2}+z\right) - \psi^{(0)}\left(\frac{1}{2}-z\right) = \pi\tan(\pi z)$$

$$6.\ \psi^{(0)}(1+z) = \psi^{(0)}(z) + \frac{1}{z}$$

$$7.\ \psi^{(0)}(1-z) = \psi^{(0)}(z) + \pi\cot(\pi z)$$

Solution 3 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^\infty \frac{x^2+\sin(\pi x)}{1+e^{\pi x}}dx=\int_0^\infty \frac{x^2}{1+e^{\pi x}}dx+\int_0^\infty \frac{\sin(\pi x)}{1+e^{\pi x}}dx=M+K$$

$$M=\int_0^\infty \frac{x^2}{1+e^{\pi x}}dx\overset{\{\pi x\to x\}}{\triangleq}\frac{1}{\pi^3}\int_0^\infty \frac{x^2}{1+e^x}dx=\frac{1}{\pi^3}\int_0^\infty \frac{x^2e^{-x}}{1+e^{-x}}dx=$$

$$\frac{1}{\pi^3}\sum_{n=0}^\infty (-1)^n\int_0^\infty e^{-x(n+1)}x^2dx\underset{\left\{\frac{dx}{dt}=\frac{1}{n+1}\right\}}{\overset{\{x(n+1)=t\}}{=}}\frac{1}{\pi^3}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^3}\int_0^\infty e^{-t}t^2dt=$$

$$\frac{1}{\pi^3}\Gamma(3)\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^3}=\frac{2}{\pi^3}\eta(3)=\frac{3}{2\pi^3}\zeta(3)$$

$$K=\int_0^\infty \frac{\sin(\pi x)}{1+e^{\pi x}}dx\overset{\{\pi x\to x\}}{\triangleq}\frac{1}{\pi}\int_0^\infty \frac{\sin(x)}{1+e^x}dx=\frac{1}{\pi}\mathcal{I}\int_0^\infty \frac{e^{ix}}{1+e^x}dx=$$

$$\frac{1}{\pi}\mathcal{I}\int_0^\infty \frac{e^{x(i-1)}}{e^{-x}+1}dx=\frac{1}{\pi}\mathcal{I}\left\{\sum_{n=0}^\infty (-1)^n\int_0^\infty e^{-x(n+1-i)}dx\right\}=\frac{1}{\pi}\mathcal{I}\left\{\sum_{n=0}^\infty \frac{(-1)^n}{n+1-i}\right\}=$$

$$\frac{1}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2+1}=-\frac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n^2+1}=\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n^2+1}-\frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{4n^2+1}=$$

$$\frac{1}{2\pi}(\pi\coth(\pi)-1)-\frac{2}{\pi}\left(\frac{\pi}{4}\coth\left(\frac{\pi}{2}\right)-\frac{1}{2}\right)=\frac{1}{2}\left(\coth(\pi)-\coth\left(\frac{\pi}{2}\right)\right)+\frac{1}{2\pi}=$$

$$\frac{1}{2}\left(\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}-\frac{e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}}{e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}}\right)+\frac{1}{2\pi}=\frac{1}{2}\left(\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}-\frac{e^\pi+e^{-\pi}+2}{e^\pi-e^{-\pi}}\right)+\frac{1}{2\pi}=$$

$$-\frac{1}{e^\pi-e^{-\pi}}+\frac{1}{2\pi}=-\frac{\operatorname{csch}(\pi)}{2}+\frac{1}{2\pi}$$

$$\Omega=M+K=\frac{3}{2\pi^3}\zeta(3)-\frac{\operatorname{csch}(\pi)}{2}+\frac{1}{2\pi}=\frac{\pi^2+3\zeta(3)-\pi^3\operatorname{csch}(\pi)}{2\pi^3}$$

Solution 4 by Nelson Javier Villaherrera Lopez-El Salvador

$$I=\int_0^\infty \frac{x^2+\operatorname{sen}(\pi x)}{1+\exp(\pi x)}dx=\int_0^\infty \frac{x^2+\operatorname{sen}(\pi x)}{1+e^{\pi x}}.\frac{e^{-\pi x}}{e^{-\pi x}}dx=\int_0^\infty \frac{x^2+\operatorname{sen}(\pi x)}{e^{-\pi x}+1}e^{-\pi x}dx=$$

$$\int_0^\infty [x^2+\operatorname{sen}(\pi x)]\frac{e^{-\pi x}}{1+e^{-\pi x}}dx=\int_0^\infty [x^2+\operatorname{sen}(\pi x)]\sum_{k=1}^\infty (-1)^{k-1}e^{-\pi xk}dx=$$

$$\int_0^\infty \sum_{k=1}^\infty (-1)^{k-1}[x^2+\operatorname{sen}(\pi x)]e^{-\pi xk}dx=\sum_{k=1}^\infty (-1)^{k-1}\int_0^\infty [x^2+\operatorname{sen}(\pi x)]e^{-\pi xk}dx=$$

$$\sum_{k=1}^\infty (-1)^{k-1}L[x^2+\operatorname{sen}(\pi x)]_{J=\pi k}=\sum_{k=1}^\infty (-1)^{k-1}\left[\frac{2!}{(\pi k)^3}+\frac{\pi}{(\pi k)^2+\pi^2}\right]=$$

$$=\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{2!}{\pi^3k^3}+\frac{\pi}{\pi^2k^2+\pi^2}\right]$$

$$=\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{2}{\pi^3k^3}+\frac{\pi}{\pi^2(k^2+1)}\right]=\sum_{k=1}^{\infty}\left[\frac{2.(-1)^{k-1}}{\pi^3k^3}+\frac{1}{\pi}\frac{(-1)^{k-1}}{k^2-j^2}\right]=$$

$$\frac{2}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-j)(k+j)}=\frac{2}{\pi^3}\left[\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{2}{(2k)^3}\right]+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(j2+k-k)}{j2(k-j)(k+j)}=$$

$$\frac{2}{\pi^3}(\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{2}{2^3k^3})+\frac{1}{j2\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(j+k-k+j)}{(k-j)(k+j)}=\frac{2}{\pi^3}(\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{2}{2^3k^3})+$$

$$\frac{1}{j2\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k-j}-\frac{1}{k+j}\right)=\frac{2}{\pi^3}(\sum_{k=1}^{\infty}\frac{1}{k^3}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^3})+\frac{1}{j2\pi}\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{k-j}-\frac{(-1)^{k-1}}{k+j}\right)=$$

$$\frac{2}{\pi^3}\cdot\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^3}+\frac{1}{j2\pi}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k-j}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+j}\right]=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}(\left[\sum_{k=1}^{\infty}\frac{1}{k-j}-\sum_{k=1}^{\infty}\frac{2}{2k-j}\right]-$$

$$-\left[\sum_{k=1}^{\infty}\frac{1}{k+j}-\sum_{k=1}^{\infty}\frac{2}{2k+j}\right])=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}(\sum_{k=1}^{\infty}\frac{1}{k-j}-\sum_{k=1}^{\infty}\frac{1}{k+j}-(\sum_{k=1}^{\infty}\frac{1}{k-\frac{j}{2}}-\sum_{k=1}^{\infty}\frac{1}{k+\frac{j}{2}}))=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left\{-\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k-j}\right)+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+j}\right)-(-\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k-\frac{j}{2}}\right)+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+\frac{j}{2}}\right))\right\}$$

$$=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left[\gamma+\psi(j+1)-\gamma-\psi(-j+1)+\gamma+\psi\left(-\frac{j}{2}+1\right)-\gamma-\psi(\frac{j}{2}+1)\right]=\frac{3\zeta(3)}{2\pi^3}+$$

$$\left[\psi(j+1)-\psi(1-j)+\psi\left(1-\frac{j}{2}\right)-\psi(\frac{j}{2}+1)\right]$$

$$=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}(\frac{1}{j}+\psi(j)-\psi(1-j)+\psi\left(1-\frac{j}{2}\right)-$$

$$-\frac{1}{\frac{j}{2}}-\psi(\frac{j}{2}))=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}(\frac{1}{j}-\frac{2}{j}+D_x\{ln[\Gamma(x)\Gamma(1-x)]\}_{x=j}-\psi\left(\frac{j}{2}\right)+\psi\left(1-\frac{j}{2}\right))=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left|-\frac{1}{j}+D_x\left\{ln\left[\frac{\pi}{sen(\pi x)}\right]\right\}\right|_{x=j}-D_y\{ln[\Gamma(y)\Gamma(1-y)]\}_{y=\frac{j}{2}})=\frac{3\zeta(3)}{2\pi^3}+$$

$$\frac{1}{j2\pi}\left[D_x\{\ln(\pi)-ln[sen(\pi x)]\}_{x=j}-D_y\left\{\ln\left(\frac{\pi}{sen(\pi y)}\right)\right\}_{y=\frac{j}{2}}-\frac{1}{j}\right]=\frac{3\zeta(3)}{2\pi^3}+$$

$$\frac{1}{j2\pi}\left[-\pi\frac{\cos(\pi j)}{\text{sen}(\pi j)}-D_y\{\ln(\pi)-ln[sen(\pi y)]\}_{y=\frac{j}{2}}-\frac{1}{j}\right]=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}(-\pi\frac{\cos(\pi j)}{\text{sen}(\pi j)}+$$

$$+\pi\frac{\cos\left(\pi\frac{j}{2}\right)}{\text{sen}\left(\pi\frac{j}{2}\right)}-\frac{1}{j})=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left\{\pi\left[\frac{2cos^2\left(\pi\frac{j}{2}\right)}{2sin\left(\pi\frac{j}{2}\right)cos\left(\pi\frac{j}{2}\right)}-\frac{\cos(\pi j)}{\text{sen}(\pi j)}\right]-\frac{1}{j}\right\}=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left\{\pi\left[\frac{1+\cos(\pi j)}{sen(\pi j)}-\frac{\cos(\pi j)}{\text{sen}(\pi j)}\right]-\frac{1}{j}\right\}=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{j2\pi}\left[\frac{\pi}{sen(\pi j)}-\frac{1}{j}\right]=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{2\pi}\left[\frac{\pi}{(e^{-\pi}-e^{\pi})/2}-\frac{1}{-1}\right]=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{2\pi}\left[\frac{\pi}{(e^{-\pi}-e^{\pi})/2}+1\right]$$
$$=\frac{3\zeta(3)}{2\pi^3}+\frac{1}{2\pi}\left[1-\frac{\pi}{sen\,\mathrm{h}(\pi)}\right]$$

2382. Find:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(x)sin^2(2y)sin^3(3z)}{xy^2z^3}dxdydz$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(x)sin^2(2y)sin^3(3z)}{xy^2z^3}dxdydz$$
$$=\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx.\int_{-\infty}^{\infty}\frac{sin^2(2y)}{y^2}dy.\int_{-\infty}^{\infty}\frac{sin^3(3z)}{z^3}dz$$
$$\{*\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx.\int_{-\infty}^{\infty}\frac{sin^2(2y)}{y^2}dy.\int_{-\infty}^{\infty}\frac{sin^3(3z)}{z^3}dz=M\,.\,N\,.\,K*\}$$
$$M=\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx=2.\int_0^{\infty}\frac{\sin(x)}{x}dx=2.\frac{\pi}{2}=\pi$$
$$N=\int_{-\infty}^{\infty}\frac{sin^2(2y)}{y^2}dy=2\int_0^{\infty}\frac{sin^2(2y)}{y^2}dy=4.\frac{\pi}{2}=2\pi$$
$$K=\int_{-\infty}^{\infty}\frac{sin^3(3z)}{z^3}dz$$
$$=2\int_0^{\infty}\frac{sin^3(3z)}{z^3}dz\overset{3z=t,\frac{dt}{dz}=3}{=}18\int_0^{\infty}\frac{sin^3(t)}{t^3}dt=\frac{81}{4}\int_0^{\infty}\frac{\sin(3t)}{t}dt-$$
$$-\frac{27}{4}\int_0^{\infty}\frac{\sin(t)}{t}dt=\frac{27}{4}(3.\frac{\pi}{2}-\frac{\pi}{2})=\frac{27}{4}\pi$$
$$\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx.\int_{-\infty}^{\infty}\frac{sin^2(2y)}{y^2}dy.\int_{-\infty}^{\infty}\frac{sin^3(3z)}{z^3}dz=M\,.\,N\,.\,K=\pi.2\pi.\frac{27}{4}\pi=\frac{27}{2}\pi^3$$

2383. Prove that:

$$\mathrm{I}=\int_0^1\frac{\log^2(x+1)}{x(x+1)^2}dx=\frac{\zeta(3)}{4}-\frac{\log^3(2)}{3}+\frac{\log^2(2)}{2}+\log(2)-1$$

Proposed by Cosghun Memmedov,Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1 \frac{\log^2(x+1)}{x(x+1)^2}dx=\int_0^1 \frac{\log^2(x+1)}{x}dx-\int_0^1 \frac{\log^2(x+1)}{x+1}dx-\int_0^1 \frac{\log^2(x+1)}{(x+1)^2}dx$$

$$=I_1-I_2-I_3$$

$$I_1=\int_0^1 \frac{\log^2(x+1)}{x}dx \overset{IBP}{=} -2\int_0^1 \frac{\log(x)\log(x+1)}{x+1}dx=\frac{\zeta(3)}{4}$$

$$I_2=\int_0^1 \frac{\log^2(x+1)}{x+1}dx=\frac{\log^3(2)}{3}$$

$$I_3=\int_0^1 \frac{\log^2(x+1)}{(x+1)^2}dx=1-\log(2)-\frac{\log^2(2)}{2}$$

$$I=I_1-I_2-I_3=\frac{\zeta(3)}{4}-\frac{\log^3(2)}{3}+\frac{\log^2(2)}{2}+\log(2)-1$$

2384. Find:

$$\int_1^\infty \frac{\ln(\sqrt{x})ln^2(1+x)}{(1+x)^2}dx$$

Proposed by Shirvan Tahirov, Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\frac{1}{2}\int_0^\infty \frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx-\frac{1}{2}\int_0^1 \frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega=\int_0^\infty \frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx \overset{\{1+x=t\}}{\cong} \int_1^\infty \frac{\ln(t-1)ln^2(t)}{t^2}dt \overset{\{\frac{1}{t}\to t\}}{\cong} \int_0^1 \frac{\ln\left(\frac{1-t}{t}\right)ln^2(t)}{\frac{1}{t^2}\cdot t^2}dt$$

$$=$$

$$\int_0^1 \ln\left(\frac{1-t}{t}\right)ln^2(t)dt=-\sum_{n=1}^\infty \frac{1}{n}\int_0^1 t^n ln^2(t)dt-\int_0^1 ln^3(t)dt=$$

$$2\sum_{n=1}^\infty \frac{1}{n(n+1)^3}+6=2\zeta(3)+\frac{\pi^3}{3}$$

$$\Omega_2=\int_0^1 \frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx \overset{\{1+x=t\}}{\cong} \int_1^2 \frac{\ln(t-1)ln^2(t)}{t^2} \overset{\{\frac{1}{t}\to t\}}{\cong} \int_{\frac{1}{2}}^1 \frac{\ln\left(\frac{1-t}{t}\right)ln^2(t)}{\frac{1}{t^2}\cdot t^2}dt=$$

$$\underbrace{\int_0^1 \ln\left(\frac{1-t}{t}\right) ln^2(t)\,dt}_{\Omega_1} - \int_0^{\frac{1}{2}} \ln\left(\frac{1-t}{t}\right) ln^2(t)\,dt = 2\zeta(3) + \frac{\pi^3}{3} - K$$

$$K = \int_0^{\frac{1}{2}} \ln(1-t)\, ln^2(t)dt - \underbrace{\int_0^{\frac{1}{2}} ln^3(t)\,dt}_{IBP} = -\sum_{n=1}^{\infty} \frac{1}{n}\int_0^{\frac{1}{2}} t^n ln^2(t)dt -$$

$$-[6tln(t) - 3tln^2(t) + tln^3(t) - 6t]_0^{\frac{1}{2}} = -\frac{1}{2}ln^2(2)\sum_{n=1}^{\infty}\frac{1}{n(n+1)2^n} + 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^{\frac{1}{2}} t^n \ln(t)\,dt +$$

$$+3\ln(2) + \frac{3ln^2(2)}{2} + \frac{ln^3(2)}{2} + 3$$

$$= -\ln(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2 2^n}}_{Partial\ Sum} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3 2^n}}_{Partial\ Sum} + ln^2(2) +$$

$$+ln^3(2) + 3\ln(2) + 3 = 2\ln(2) + \frac{7\zeta(3)}{4} + \frac{ln^3(2)}{3} + \frac{\pi^2}{6} + ln^2(2)$$

$$\Omega_2 = \Omega_1 - K\,,\ \ \Omega = \frac{1}{2}(\Omega_1 - \Omega_2) = \frac{1}{2}K = \frac{7\zeta(3)}{8} + \frac{\pi^2}{12} + \frac{ln^3(2)}{6} + \frac{ln^2(2)}{2} + \ln(2)$$

$Note:\ \zeta(3) \to Apery's\ constant$

2385. Find:

$$\Omega = \lim_{n\to\infty}\left(logn + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n - 1}}{x^2}\right)$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(\log(n) + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right) = \lim_{n\to 0}(\log(n) + f(n))$$

$$f(n) = \lim_{x\to 0}\frac{1+x^2-(1+x^2)^{H_n}}{x^2(1+x^2)}\quad \{1+x^2 = t\ ,\qquad x^2 = t-1\}$$

$$f(n) = \lim_{t\to 1}\frac{t-t^{H_n}}{t^2-t} = \lim_{x\to 1}\frac{\frac{\partial}{\partial t}(t-t^{H_n})}{\frac{\partial}{\partial t}(t^2-t)} = \lim_{t\to 1}\frac{1-H_n t^{H_n-1}}{2t-1} = 1-H_n$$

$$\Omega = \lim_{n\to 0}(\log(n) - H_n + 1) = \lim_{n\to\infty}\left(\log(n) - \ln(n) - \gamma - \frac{1}{2n} + \xi_n + 1\right) = 1-\gamma$$

$$\left\{0 \le \xi_n \le \frac{1}{8n^2}\ \ n\to\infty\ \ \xi_n \to 0\right\}$$

2386. ***Prove that***

$$\int_0^1 \frac{\ln(x)\big(1+\ln(x)+\ln(1-x)\big)}{1+x^2}dx = \frac{1}{64}\left(7\pi^3+4\pi ln^2(2)-64G-64Im\{Li_3(1+i)\}\right)$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \frac{\ln(x)\big(1+\ln(x)+\ln(1-x)\big)}{1+x^2}dx$$

$$= \int_0^1 \frac{\ln(x)}{1+x^2}dx + \int_0^1 \frac{ln^2(x)}{1+x^2}dx + \int_0^1 \frac{\ln(x)\ln(1-x)}{1+x^2}dx = \Omega_1+\Omega_2+\Omega_3$$

Note:

$$\left\{\int_0^1 \frac{\ln^k(x)}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{2n}\ln^k(x)\,dx = (-1)^k k!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{k+1}} = (-1)^k k!\,\beta(k+1)\right\}$$

$$\Omega_1 = \int_0^1 \frac{\ln(x)}{1+x^2}dx = -\beta(2) = -G;\ \Omega_2 = \int_0^1 \frac{ln^2(x)}{1+x^2}dx = 2\beta(3) = \frac{\pi^3}{16}$$

$$\Omega_3 = \int_0^1 \frac{\ln(x)\ln(1-x)}{1+x^2}dx,\ \Omega_3(a) = \int_0^1 \frac{\ln(x)\ln(1-ax)}{1+x^2}dx,\ \{\Omega_3(1)=\Omega_3, \Omega_3(0)=0\}$$

$$\frac{d}{da}\Omega_3(a) = \frac{d}{da}\int_0^1 \frac{\ln(x)\ln(1-ax)}{1+x^2}dx = \int_0^1 \frac{x\ln(x)}{(ax-1)(1+x^2)}dx =$$

$$= \frac{a}{1+a^2}\int_0^1 \frac{\ln(x)}{1+x^2}dx - \frac{1}{1+a^2}\int_0^1 \frac{x\ln(x)}{1+x^2}dx - \frac{a}{1+a^2}\int_0^1 \frac{\ln(x)}{1-ax}dx =$$

$$= -\frac{a\beta(2)}{1+a^2} + \frac{1}{1+a^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2} - \frac{a}{1+a^2}\sum_{n=0}^{\infty}a^n\int_0^1 x^n\ln(x)dx =$$

$$= -\frac{aG}{1+a^2} + \frac{\eta(2)}{4(1+a^2)} + \frac{a}{1+a^2}\sum_{n=0}^{\infty}\frac{a^n}{(n+1)^2} = -\frac{aG}{1+a^2} + \frac{\pi^2}{48(1+a^2)} + \frac{Li_2(a)}{1+a^2}$$

$$\Omega_3 = \frac{\pi^2}{48}\int_0^1 \frac{1}{1+a^2}da + \int_0^1 \frac{Li_2(a)}{1+a^2}da - G\int_0^1 \frac{a}{1+a^2}da = \frac{\pi^2}{48}I_1 + I_2 - GI_3$$

$$I_1 = \int_0^1 \frac{1}{1+a^2}da = [\arctan(a)]_0^1 = \arctan(1) = \frac{\pi}{4}$$

$$I_2 = \int_0^1 \frac{Li_2(a)}{1+a^2}da = \sum_{n=0}^{\infty}(-1)^n\int_0^1 a^{2n}Li_2(a)da$$

Notes:

$$\left\{\begin{array}{c}\int_0^1 x^{n-1}Li_2(x)dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\\ \sum_{n=1}^{\infty}\frac{x^n}{n^2}H_n=Li_3(x)-Li_3(1-x)+\ln(1-x)Li_2(1-x)+\frac{1}{2}\ln(x)ln^2(1-x)+\zeta(3),\qquad |x|\le 1\end{array}\right\}$$

$$I_2=\sum_{n=0}^{\infty}(-1)^n\int_0^1 a^{2n+1-1}Li_2(a)da=\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2)}{2n+1}-\frac{H_{2n+1}}{(2n+1)^2}\right]=$$

$$=\zeta(2)\arctan(1)-\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}}{(2n+1)^2}=\frac{\pi^3}{24}-\dot{\imath}m\left\{\sum_{n=1}^{\infty}\frac{i^nH_n}{n^2}\right\}$$

Note: $\left\{\sum_{n=0}^{\infty}(-1)^nf(2n+1)=\dot{\imath}m\{\sum_{n=1}^{\infty}i^nf(n)\}\right\}$

$$I_2=\frac{\pi^3}{24}-\dot{\imath}m\left\{Li_3(i)-Li_3(1-i)+\ln(1-i)\,Li_2(1-i)+\frac{1}{2}\ln(i)\,ln^2(1-i)+\zeta(3)\right\}$$

Notes:

$$\ln(1-i)=\ln\left(\sqrt{2}\left(cos\left(\frac{\pi}{4}\right)-isin\left(\frac{\pi}{4}\right)\right)\right)=\frac{1}{2}\ln(2)+\ln\left(e^{\frac{-i\pi}{4}}\right)=\frac{1}{2}\ln(2)-\frac{i\pi}{4}$$

$$\ln(i)=\ln\left(cos\left(\frac{\pi}{2}\right)+isin\left(\frac{\pi}{2}\right)\right)=\ln\left(e^{\frac{i\pi}{2}}\right)=\frac{i\pi}{2}$$

$$Li_2(1-i)=\frac{3}{8}\zeta(2)-i\left(\frac{\pi}{4}\ln(2)+G\right),Li_3(i)=\frac{i\pi^3}{32}-\frac{3\zeta(3)}{32}$$

$$I_2=\frac{\pi^3}{24}+\frac{\pi}{16}ln^2(2)+\frac{G}{2}\ln(2)-\dot{\imath}m\{Li_3(1+i)\}$$

$$I_3=\int_0^1\frac{a}{1+a^2}da,\left\{\tan^{-1}(a)=t,\ dt=\frac{da}{1+a^2},t\left[\frac{\pi}{4},0\right],a=\tan(t)\right\}$$

$$I_3=\int_0^{\frac{\pi}{4}}\tan(t)dt=-[\ln(cost)]_0^{\frac{\pi}{4}}=\frac{\ln(2)}{2}$$

$$\Omega_3=\frac{\pi^2}{48}I_1+I_2-GI_3=\frac{\pi^3}{192}+\frac{\pi^3}{24}-\dot{\imath}m\{Li_3(1+i)\}+\frac{\pi}{16}ln^2(2)+\frac{G}{2}\ln(2)-\frac{G}{2}\ln(2)=$$

$$=\frac{3\pi^3}{64}+\frac{\pi}{16}ln^2(2)-\dot{\imath}m\{Li_3(1+i)\}$$

$$\int_0^1\frac{\ln(x)(1+\ln(x)+\ln(1-x))}{1+x^2}dx=\Omega_1+\Omega_2+\Omega_3$$

$$=\frac{3\pi^3}{64}+\frac{\pi}{16}ln^2(2)-G-\dot{\imath}m\{Li_3(1+i)\}+\frac{\pi^3}{16}=$$

$$=\frac{1}{64}(7\pi^3+4\pi ln^2(2)-64G-64\dot{\imath}m\{Li_3(1+i)\}),\textit{Hence proved}$$

2387. ***Prove that:***

$$\int_0^{\infty} e^{-x^2}\sqrt{(\cosh^2(x)-1)}\,dx = \frac{e^{\frac{1}{4}}}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Amin Hajiyev-Azerbaijan

$$\sigma = \int_0^{\infty} e^{-x^2}\sqrt{\cosh^2(x)-1}\,dx = \int_0^{\infty} e^{-x^2}\sinh(x)\,dx$$

$$= \int_0^{\infty} \frac{e^{-x^2}}{2}(e^x - e^{-x})dx = \frac{1}{2}(\int_0^{\infty} e^{-x^2+x}\,dx -$$

$$-\int_0^{\infty} e^{-x^2-x}dx) = \frac{1}{2}(\int_0^{\infty} e^{-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx - \int_0^{\infty} e^{-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}dx) =$$

$$= \frac{1}{2}e^{\frac{1}{4}}\left(\int_0^{\infty} e^{-\left(x-\frac{1}{2}\right)^2}dx - \int_0^{\infty} e^{-\left(x+\frac{1}{2}\right)^2}dx\right) = \frac{e^{\frac{1}{4}}}{2}(\sigma_1 - \sigma_2)$$

$$\sigma_1 = \int_0^{\infty} e^{-\left(x-\frac{1}{2}\right)^2}dx = \frac{\sqrt{\pi}}{2}[\,\mathrm{erf}\left(x-\frac{1}{2}\right)]_0^{\infty} = \frac{\sqrt{\pi}}{2}\left[\mathrm{erf}(\infty) - \mathrm{erf}\left(\frac{1}{2}\right)\right] = \frac{\sqrt{\pi}}{2}\left(1+\mathrm{erf}\left(\frac{1}{2}\right)\right)$$

$$\sigma_2 = \int_0^{\infty} e^{-\left(x+\frac{1}{2}\right)^2}dx = \frac{\sqrt{\pi}}{2}[\mathrm{erf}\left(x+\frac{1}{2}\right)]_0^{\infty} = \frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\,(\infty) - \mathrm{erf}\,\left(\frac{1}{2}\right)\right) = \frac{\sqrt{\pi}}{2}\left(1-\mathrm{erf}\,\left(\frac{1}{2}\right)\right)$$

$$\sigma = \int_0^{\infty} e^{-x^2}\sqrt{\cosh^2(x)-1}dx = \frac{e^{\frac{1}{4}}}{2}(\sigma_1-\sigma_2) = \frac{e^{\frac{1}{4}}\sqrt{\pi}}{2}\mathrm{erf}\left(\frac{1}{2}\right)$$

Notes:

Error function: $\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}dz = \mathrm{erf}\,[z]$; $\mathrm{erf}(\pm\infty) = \pm 1$; $\mathrm{erf}\,(\pm a) = \pm\,\mathrm{erf}\,(a)$

2388. **Prove that :**

$$\int_0^1 \frac{x}{\sinh(x)}dx = Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln(\frac{e+1}{e-1}) + \frac{\pi^2}{4}$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\omega = \int_0^1 \frac{x}{\sinh(x}dx = 2\int_0^1 \frac{x\,e^x}{e^{2x}-1}dx \quad \left\{e^{-x} = t;\ x = -\ln(t)\,;\frac{dx}{dt} = -\frac{1}{t}\right\}$$

$$\omega = 2\int_1^{\frac{1}{e}} \frac{\frac{1}{t}\ln(t)}{(\frac{1}{t^2}-1)t}dt = 2\int_1^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt =$$

$$= 2\left(\int_0^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt - \int_0^1 \frac{\ln(t)}{1-t^2}dt\right) = 2(\omega_1 - \omega_2)$$

ω_1

$$= \int_0^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt = \sum_{n=0}^{\infty}\int_0^{\frac{1}{e}} t^{2n}\ln(t)dt \underset{\text{İBP}}{=} \sum_{n=0}^{\infty}\left(-\frac{1}{(2n+1)e^{2n}+1} - \frac{1}{(2n+1)^2e^{2n+1}}\right) =$$

$$= -\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{ne^n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{ne^n} + \sum_{n=1}^{\infty}\frac{1}{n^2e^n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2e^n}\right) =$$

$$= -\frac{1}{2}\left(1 - \ln(e-1) - 1 + \ln(e+1) + Li_2\left(\frac{1}{e}\right) - Li_2\left(-\frac{1}{e}\right)\right)$$

$$= -\frac{1}{2}\ln\left(\frac{e+1}{e-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{e}\right) + \frac{1}{2}Li_2\left(-\frac{1}{e}\right)$$

$$\omega_2 = \int_0^1 \frac{\ln(t)}{1-t^2}dt =$$

$$= \sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln(t) = -\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \underset{\text{İBP}}{=} -\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right) =$$

$$= -\frac{1}{2}\left(\zeta(2) - \frac{1}{2}\zeta(2)\right) = -\frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = -\frac{\pi^2}{8}\ ;$$

$$\omega = 2(\omega_1 - \omega_2) = 2\left(-\frac{1}{2}\ln\left(\frac{e+1}{e-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{e}\right) + \frac{1}{2}Li_2\left(-\frac{1}{e}\right) + \frac{\pi^2}{8}\right)$$

$$= Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln\left(\frac{e+1}{e-1}\right) + \frac{\pi^2}{4}$$

$$\int_0^1 \frac{x}{sinh(x)}dx = Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln\left(\frac{e+1}{e-1}\right) + \frac{\pi^2}{4}$$

Notes:

$$\sum_{n=0}^{\infty} a_{2n+1} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty}(-1)^n a_n\right);\ sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$Li_s(n) = \sum_{n=1}^{\infty}\frac{x^n}{n^s}\ \ Polylogarithm\ function$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\ \ Rieman's\ zeta\ function$$

2389. ***Prove that :***

$$\int_{R^+}\left(\frac{\sin(x)}{\sinh(x)}\right)\left(\frac{\cos(x)}{\cosh(x)}\right)dx=\frac{\pi}{4}\tanh\left(\frac{\pi}{2}\right)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\sigma=\int_0^\infty\frac{\sin(x)\cos(x)}{\sinh(x)\cosh(x)}dx=\int_0^\infty\frac{\sin(2x)}{\sinh(2x)}dx=\frac{1}{2}\int_0^\infty\frac{\sin(x)}{\sinh(x)}dx=\int_0^\infty\frac{e^{-x}\sin(x)}{1-e^{-2x}}dx$$

$$=\int_0^\infty\sum_{n=0}^\infty e^{-2nx-x}\sin(x)\,dx=\sum_{n=0}^\infty\int_0^\infty e^{-x(2n+1)}\sin(x)dx=\sum_{n=0}^\infty(f(2n+1))$$

$$f(a)=\int_0^\infty e^{-ax}\sin(x)dx,\quad Using\ IBP\ method\ \left\{u=\sin(x),du=\cos(x)\,dx;v=\int e^{-ax}dx=-\frac{1}{a}e^{-ax}\right\}$$

$$f(a)=[-\frac{e^{-ax}\sin(x)}{a}]_0^\infty+\frac{1}{a}\int_0^\infty e^{-ax}\cos(x)dx=\frac{1}{a}[-\frac{e^{-ax}\cos(x)}{a}]_0^\infty-\frac{1}{a^2}\int_0^\infty e^{-ax}\sin(x)\,dx$$

$$f(a)=\frac{1}{a^2}-\frac{1}{a^2}f(a)\,;\ f(a)=\frac{1}{a^2+1},f(2n+1)=\frac{1}{1+(2n+1)^2}$$

$$\sigma=\sum_{n=0}^\infty f(2n+1)=\sum_{n=0}^\infty\frac{1}{(2n+1)^2+1}=\sum_{n=0}^\infty\frac{1}{(2n+1-i)(2n+1+i)}$$

$$=\frac{1}{4}\sum_{n=0}^\infty\frac{1}{\left(n+\frac{1-i}{2}\right)\left(n+\frac{1+i}{2}\right)}=$$

$$=\frac{1}{4}\left(\frac{\psi^{(0)}\left(\frac{1+i}{2}\right)-\psi^{(0)}\left(\frac{1-i}{2}\right)}{i}\right)=\frac{1}{4i}\left(\psi^{(0)}\left(\frac{1+i}{2}\right)-\psi^{(0)}\left(1-\frac{1+i}{2}\right)\right)=$$

$$=-\frac{1}{4i}\left(\pi\cot\left(\pi\frac{1+i}{2}\right)\right)=\frac{\pi}{4i}\tan\left(\frac{\pi i}{2}\right)=\frac{\pi}{4}tanh\left(\frac{\pi}{2}\right)$$

Notes:

$$Polygamma\,reflection\,formula:\ (-1)^m\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\frac{d^m}{dz^m}\cot z)$$

Hiperbolic tangent function:

$$tanh(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}};\ tan(x)=\frac{i(e^{-ix}-e^{ix})}{e^{-ix}+e^{ix}};\ tanh(ix)=i\,tan(x)$$

2390. Prove that:

$$\int_0^\infty\left(\frac{\sin(x)}{\sinh(x)}\right)\left(\frac{\cos(x)}{\cosh(x)}\right)\left(\frac{\tan(x)}{\tanh(x)}\right)dx=\frac{\pi}{2}\coth(\pi)-\frac{1}{2}$$

Proposed by Ankush Kumar Parcha-India

Solution by Cosghun Memmedov-Azerbaijan

$$\int_0^{\infty}\left(\frac{\sin(x)}{\sinh(x)}\right)\left(\frac{\cos(x)}{\cosh(x)}\right)\left(\frac{\tan(x)}{\tanh(x)}\right)dx=$$

$$\int_0^{\infty}\frac{4\sin^2(x)}{e^{2x}+e^{-2x}-2}dx=\int_0^{\infty}\frac{2-2\cos(2x)}{e^{2x}+e^{-2x}-2}dx\overset{(2x\to x)}{\cong}\int_0^{\infty}\frac{1-\cos(x)}{e^{x}+e^{-x}-2}dx=$$

$$\int_0^{\infty}\frac{e^{-x}}{(1-e^{-x})^2}dx-\int_0^{\infty}\frac{e^{-x}\cos(x)}{(1-e^{-x})^2}dx=$$

$$\left(\sum_{n=0}^{\infty}e^{-xn}=\frac{1}{1-e^{-x}};\ \frac{\partial}{\partial x}\sum_{n=0}^{\infty}e^{-xn}=-\frac{e^{-x}}{(1-e^{-x})^2}\Rightarrow\sum_{n=1}^{\infty}ne^{-xn}=\frac{e^{-x}}{(1-e^{-x})^2}\right)$$

$$=\sum_{n=1}^{\infty}n\int_0^{\infty}e^{-xn}dx-\Re\left(\sum_{n=1}^{\infty}n\int_0^{\infty}e^{-x(n-i)}dx\right)=\sum_{n=1}^{\infty}n\times\frac{1}{n}-\Re\left(\sum_{n=1}^{\infty}\frac{n}{n-i}\right)=$$

$$\sum_{n=1}^{\infty}\left(1-\frac{n^2}{n^2+1}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{\pi}{2}\coth(\pi)-\frac{1}{2}$$

Notes ; $\sum_{-\infty}^{+\infty}\frac{1}{n^2+a^2}=\frac{\pi}{a}\coth(\pi)$; $\sinh(x)=\frac{e^x-e^{-x}}{2}$; $\cosh(x)=\frac{e^x+e^{-x}}{2}$

2391. Find:

$$\Omega_{n,p,t}=\int_a^b\frac{\ln^n(x+1)}{x^p(x+1)^t}dx\quad\{0\le p,t,n\in Z\text{ and }0\le a<b\}$$

Proposed by Bui Hong Suc-Vietnam *(Inspired by : Shirvan Tahirov)*

Solution by Noureddine Sima-Algeria

$$\int_a^b\frac{\ln^n(x+1)}{x^p(x+1)^t}dx\ \underset{dx=e^z dz}{\overset{1+x=e^z}{=}}\int_{\ln(1+a)}^{\ln(1+b)}\frac{z^n e^{-z(t-1+p)}}{(1-e^{-z})^p}dz$$

$$\frac{1}{(1-y)^m}=\sum_{k=m-1}^{\infty}C_k^{m-1}y^{k+1-m}$$

$$\sum_{k=p-1}^{\infty}C_k^{p-1}\int_{\ln(1+a)}^{\ln(1+b)}z^n e^{-z(t+k)}dz\overset{z=\frac{y}{t+k}}{\cong}$$

$$\sum_{k=p-1}^{\infty}\frac{C_k^{p-1}}{(t+k)^{n+1}}\int_{(t+k)\ln(1+a)}^{(t+k)\ln(1+b)}y^n e^{-y}dy\overset{e^{-y}=\sum_{i=0}^{\infty}\frac{(-1)^i y^i}{i!}}{\cong}$$

$$\sum_{k=p-1}^{\infty}\frac{C_k^{p-1}}{(t+k)^{n+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_{(t+k)\ln(1+a)}^{(t+k)\ln(1+b)}y^{n+i}dy=$$

$$\sum_{k=p-1}^{\infty}\frac{C_k^{p-1}}{(t+k)^{n+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\,(n+i+1)}(t+k)^{n+i+1}[ln^{n+i+1}(1+b)-ln^{n+i+1}(1+a)]$$

2392. ***Prove that:***

$$\int_0^1\frac{ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\frac{1}{64}\left(42\zeta(3)+16G+2\pi^3+\pi^2-8\pi\ln(2)\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\theta=\int_0^1\frac{ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)}dx+\int_0^1\frac{x\arctan(x)}{(1+x)(1+x^2)}dx=\theta_1+\theta_2$$

$$\theta_1=\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)}dx=\frac{1}{2}\left(\int_0^1\frac{ln^2(x)}{1+x}dx+\int_0^1\frac{\ln^2(x)}{1+x^2}dx-\int_0^1\frac{x\,ln^2(x)}{1+x^2}dx\right)=$$

$$=\frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n\int_0^1x^n ln^2(x)dx+\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}ln^2(x)dx-\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n+1}ln^2(x)dx\right)=$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^3}\right)=\eta(3)+\beta(3)-\frac{1}{8}\eta(3)=\frac{7}{8}\eta(3)+\frac{\pi^3}{32}$$

$$=\frac{21\zeta(3)}{32}+\frac{\pi^3}{32}$$

$$\theta_2=\int_0^1\frac{x\arctan(x)}{(1+x)(1+x^2)}dx$$

$$Using\ \dot{I}BP\ method\ \{u=\arctan(x);\frac{du}{dx}=\frac{1}{1+x^2};\ v=\int\frac{x+1-1}{(1+x)(1+x^2)}dx$$

$$=\frac{1}{2}\tan^{-1}(x)+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\ln(1+x)\}$$

$$\theta_2=(\arctan(x)\left(\frac{1}{2}\arctan(x)+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\ln(1+x)\right))_0^1$$

$$-\frac{1}{2}\int_0^1\frac{\tan^{-1}(x)}{1+x^2}dx-\frac{1}{4}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}dx=$$

$$=\frac{\pi^2}{32}+\frac{\pi\ln(2)}{16}-\frac{\pi\ln(2)}{8}-\frac{1}{2}\int_0^{\frac{\pi}{4}}x\,dx-\frac{1}{4}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}dx$$

$$=\frac{\pi^2}{64}-\frac{\pi\ln(2)}{16}-\frac{1}{4}J+\frac{1}{2}I$$

$$J=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx;\ subtitation\ \left\{\arctan(x)=t,\frac{dt}{dx}=\frac{1}{1+x^2},t\left[\frac{\pi}{4};0\right]\right\}$$

$$J=-2\int_0^{\frac{\pi}{4}}\ln(\cos(t))dt;\ \ note:\{\ Fourier\ series\ \ln(\cos(x))$$
$$=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\}$$
$$J=2\ln(2)\int_0^{\frac{\pi}{4}}dt+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nt)dt=\frac{\pi}{2}\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}=\frac{\pi}{2}\ln(2)-\mathrm{G}$$
$$I=\int_0^1\frac{\ln(1+x)}{1+x^2}dx;\ subtitation:\left\{\frac{1-t}{1+t}=x,\frac{dx}{dt}=-\frac{2}{(1+t)^2},t[0;1]\right\}$$
$$I=\int_0^1\frac{\ln\left(\frac{2}{1+t}\right)}{1+t^2}dt=\ln(2)\int_0^1\frac{1}{1+t^2}dt-\int_0^1\frac{\ln(1+t)}{1+t^2}dt=\frac{\pi}{4}\ln(2)-I;\ \ 2I=\frac{\pi}{4}\ln(2)\ \ I$$
$$=\frac{\pi}{8}\ln(2)$$
$$\theta_2=\frac{\pi^2}{64}-\frac{\pi\ln(2)}{16}-\frac{1}{4}J+\frac{1}{2}I=\frac{\pi^2}{64}-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}\ln(2)+\frac{G}{4}+\frac{\pi}{16}\ln(2)=\frac{\pi^2}{64}-\frac{\pi}{8}\ln(2)+\frac{G}{4}$$
$$\int_0^1\frac{ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\theta_1+\theta_2=\frac{21}{32}\zeta(3)+\frac{\pi^3}{32}+\frac{\pi^2}{64}-\frac{\pi}{8}\ln(2)+\frac{G}{4}$$
$$=\frac{1}{64}\left(42\zeta(3)+2\pi^3+\pi^2+16G-8\pi\ln(2)\right)$$

2393. Find:

$$\int_0^1\int_1^{\infty}\frac{yln(1-y^2)ln^2(1+x)}{x^2(1+x)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pham Duc Nam-Vietnam

$$\int_0^1\int_1^{\infty}\frac{yln(1-y^2)ln^2(1+x)}{x^2(1+x)}dxdy=\int_0^1 yln(1-y^2)dy\underbrace{\int_1^{\infty}\frac{ln^2(1+x)}{x^2(1+x)}dx}_{x\to\frac{1}{x}}=$$
$$-\frac{1}{2}\int_0^1 ln(1-y^2)d(1-y^2)\int_0^1\frac{xln^2\left(1+\frac{1}{x}\right)}{1+x}dx=-\frac{1}{2}\int_0^1\frac{xln^2\left(1+\frac{1}{x}\right)}{1+x}dx=-\frac{1}{2}I$$
$$*\ Consider:\int_0^1\frac{xln^2(1+x)}{1+x}dx\overset{IBP}{=}2-\frac{1}{3}ln^3(2)+2ln^2(2)-4\ln(2)$$
$$But:\int_0^1\frac{xln^2(1+x)}{1+x}dx=\int_0^1\frac{x(\ln(x)+\ln(1+\frac{1}{x}))^2}{1+x}dx=I+\int_0^1\frac{xln^2(x)}{1+x}dx+$$

$$2\int_0^1 \frac{xln(x)\ln(1+\frac{1}{x})}{1+x}dx = I + 2\int_0^1 \frac{xln(x)\ln(1+x)}{1+x}dx - \int_0^1 \frac{xln^2(x)}{1+x}dx$$

$$1)\int_0^1 \frac{xln^2(x)}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{n+1}ln^2(x)dx = 2n$$

$$= 2\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+2)^3} = 2 - \frac{3}{2}\zeta(3)$$

$$2)\ \int_0^1 \frac{xln(x)\ln(1+x)}{1+x}dx = -\int_0^1 xln(x)\sum_{n=1}^{\infty}(-1)^n H_n x^n dx =$$

$$-\sum_{n=1}^{\infty}(-1)^n H_n \int_0^1 x^{n+1}\ln(x)\,dx = \sum_{n=1}^{\infty}(-1)^n H_n \frac{1}{(n+2)^2} =$$

$$\sum_{n=1}^{\infty}(-1)^n\left(H_{n+2} - \frac{1}{n+2} - \frac{1}{n+1}\right)\frac{1}{(n+2)^2} = \sum_{n=1}^{\infty}(-1)^n \frac{H_{n+2}}{(n+2)^2} - \underbrace{\sum_{n=1}^{\infty}(-1)^n \frac{1}{(n+2)^3}}_{=\frac{7}{8}-\frac{3}{4}\zeta(3)}$$

$$-\underbrace{\sum_{n=1}^{\infty}(-1)^n \frac{1}{(n+1)(n+2)^2}}_{=\frac{\pi^2}{12}+2\ln(2)-\frac{9}{4}} = \frac{3}{4}\zeta(3) + \frac{11}{8} - \frac{\pi^2}{12} - 2\ln(2) + \sum_{n=3}^{\infty}(-1)^n \frac{H_n}{n^2} = \frac{3}{4}\zeta(3) + \frac{11}{8} -$$

$$-\frac{\pi^2}{12} - 2\ln(2) + \left(\underbrace{\sum_{n=1}^{\infty}(-1)^n \frac{H_n}{n^2}}_{=-\frac{5}{8}\zeta(3)} + 1 - \frac{3}{8}\right) = \frac{\zeta(3)}{8} - \frac{\pi^2}{12} + 2 - 2\ln(2)$$

$$=> 2 - \frac{1}{3}ln^3(2) + 2ln^2(2) - 4\ln(2)$$

$$= I + 2\left(\frac{\zeta(3)}{8} - \frac{\pi^2}{12} + 2 - 2\ln(2)\right) - \left(2 - \frac{3}{2}\zeta(3)\right)$$

$$=> I = -\frac{7}{4}\zeta(3) + \frac{\pi^2}{6} - \frac{1}{3}ln^3(2) + 2ln^2(2)$$

$$\int_0^1\int_1^{\infty} \frac{yln(1-y^2)ln^2(1+x)}{x^2(1+x)}dxdy = -\frac{1}{2}I = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12} + \frac{1}{6}ln^3(2) - ln^2(2) =$$
$$\frac{1}{24}\left(21\zeta(3) + 4ln^3(2) - 2\pi^2 - 24ln^2(2)\right)$$

Note : $\zeta(3) \rightarrow$ Apery's constant

2394. Prove the integral relation:

$$\int_0^\infty e^{\frac{-\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx=\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}\int_0^\infty e^{\frac{-\pi 3}{3}}\sqrt[3]{\coth(3\pi x)-1}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Pham Duc Nam-Vietnam

$$*I=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\frac{6^{6\pi x}+1}{e^{6\pi x}-1}+1}dx=$$

$$=\sqrt[3]{2}\int_0^\infty e^{\frac{5\pi x}{3}}\sqrt[3]{\frac{1}{e^{6\pi x}-1}}dx$$

$$t=\sqrt[3]{\frac{1}{e^{6\pi x}-1}}\Rightarrow x=\frac{1}{6\pi}\ln(1+t^3)\Rightarrow dx=-\frac{1}{2\pi}\frac{1}{t^4+t}dt$$

$$\Rightarrow I=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\left(\frac{1}{t^3}+1\right)^{\frac{5}{18}}\frac{1}{t^3+1}dt=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\frac{t^{-\frac{5}{6}}}{(t^3+1)^{\frac{13}{18}}}dt=$$

$$=\frac{\sqrt[3]{2}}{6\pi}\int_0^\infty\frac{(t^3)^{\frac{1}{18}-1}}{(t^3+1)^{\frac{1}{18}+\frac{12}{18}}}d(t^3)=\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{1}{18},\frac{12}{18}\right)$$

$$*J=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)-1}dx=\int_0^\infty 3^{-\frac{\pi x}{3}}\sqrt[3]{\frac{e^{6\pi x}+1}{e^{6\pi x-1}}-1}dx=$$

$$=\sqrt[3]{2}\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\frac{1}{e^{6\pi x}-1}}dx$$

$$t=\sqrt[3]{\frac{1}{e^{6\pi x}-1}}\Rightarrow x=\frac{1}{6\pi}\ln\left(1+\frac{1}{t^3}\right)\Rightarrow dx=-\frac{1}{2\pi}\frac{1}{t^4+t}dt$$

$$\Rightarrow I=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\left(\frac{1}{t^3}+1\right)^{-\frac{1}{18}}\frac{1}{t^3+1}dt=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\frac{t^{\frac{1}{6}}}{(t^3+1)^{\frac{19}{18}}}dt=$$

$$=\frac{\sqrt[3]{2}}{6\pi}\int_0^\infty\frac{(t^3)^{\frac{7}{18}-1}}{(t^3+1)^{\frac{7}{18}+\frac{12}{18}}}d(t^3)=\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{7}{18},\frac{12}{18}\right)$$

$$\Rightarrow \frac{I}{J} = \frac{\sqrt[3]{\frac{2}{6\pi}}B\left(\frac{1}{18},\frac{12}{18}\right)}{\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{7}{18},\frac{12}{18}\right)} = \frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{12}{18}\right)}{\Gamma\left(\frac{13}{18}\right)}\frac{\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{12}{18}\right)} - \frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}$$

$$\Rightarrow \int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx = \frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)-1}dx$$

Hence proved.

2395. ***Find:***

$$\int_0^1 \frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\lambda = \int_0^1 \frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx = \frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)\log\left(\frac{1+x}{1-x}\right)}{1+x^2}dx$$

$$let\left\{\tan^{-1}(x) = t,\ dt = \frac{dx}{1+x^2}, t\left[\frac{\pi}{4};0\right]\right\}$$

$$\lambda = \frac{1}{2}\int_0^{\frac{\pi}{4}} t\log\left(\frac{1+\tan(t)}{1-\tan(t)}\right)dt = \frac{1}{2}\int_0^{\frac{\pi}{4}} t\log\left(\tan\left(t+\frac{\pi}{4}\right)\right)dt =$$

$$let:\left\{\theta = \left(t+\frac{\pi}{4}\right), d\theta = dt, t = \theta - \frac{\pi}{4}\ \ \theta\left[\frac{\pi}{2};\frac{\pi}{4}\right]\right\}$$

$$\lambda = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\left(\theta-\frac{\pi}{4}\right)\log(\tan(\theta))d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\theta\log(\tan(\theta))d\theta - \frac{\pi}{8}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\log(\tan(\theta))d\theta = \frac{1}{2}\lambda_1 - \frac{\pi}{8}\lambda_2$$

$$Note:\left\{Fourier\ series\ of\ \log(\tan(\theta)) = -2\sum_{n=0}^{\infty}\frac{\cos(4n\theta+2\theta)}{2n+1}\right\}$$

$$\lambda_1 = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\theta\cos(\theta(4n+2))d\theta = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\theta\sin(\theta(4n+2))}{4n+2} + \frac{\cos(\theta(4n+2))}{(4n+2)^2}\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{2\sin(\pi n)}{4(2n+1)^2} - \frac{\cos(2\pi n)}{4(2n+1)^2} - \frac{\pi\sin(\pi n)\cos(\pi n)}{2(2n+1)} - \frac{\pi\cos(\pi n)}{8(2n+1)}\right]$$

$$Notes:\{\sin(\pi n) = 0, \cos(2\pi n) = 1, \cos(\pi n) = (-1)^n, n\epsilon N\cup\{0\}\}$$

$$\lambda_1 = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{1}{4(2n+1)^2} + \frac{\pi(-1)^n}{8(2n+1)}\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} + \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$Notes:$$

$$\left\{Dirichlet\ beta\ function: \beta(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}, \beta(2) = G\ (Catalan's\,constant)\right\}$$

$$\left\{Rieman's\ zeta\ function: \zeta(z) = \frac{1}{1-2^{-z}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^z}\right\}$$

$$\lambda_1 = \frac{1}{2}(1-2^{-3})\zeta(3) + \frac{\pi G}{4} = \frac{7}{16}\zeta(3) + \frac{\pi G}{4}$$

$$\lambda_2 = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \log(\tan(\theta))d\theta = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\cos(\theta(4n+2))d\theta =$$

$$= -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\sin(\theta(4n+2))}{2(2n+1)}\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\sin(2\pi n)}{2n+1} + \frac{\cos(\pi n)}{2n+1}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = G$$

$$\int_0^1 \frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx = \frac{1}{2}\lambda_1 - \frac{\pi}{8}\lambda_2 = \frac{7}{32}\zeta(3) + \frac{\pi G}{8} - \frac{\pi G}{8} = \frac{7}{32}\zeta$$

2396.

$$Ci(x) = \gamma + \log x + \sum_{n=1}^{\infty}\frac{(-x^2)^n}{2n\cdot(2n)!}$$

Prove that:

$$\int_a^b \left(\frac{\cos x}{x}\right)^2 dx + \int_a^b Ci^2(x)\,dx \ge Ci^2(b) - Ci^2(a), \qquad 0 < a \le b$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov – Azerbaijan

$$Ci(x) = \gamma + \log(x) + \sum_{n=1}^{\infty}\frac{(-x^2)^n}{2n\cdot(2n)!}$$

$$\int_a^b \left(\frac{\cos x}{x}\right)^2 dx + \int_a^b Ci^2(x)\,dx \ge Ci^2(b) - Ci^2(a) \to 0 < a \le b$$

$$\int_a^b (f(x)^2 + f'(x)^2)\,dx \ge \int_a^b 2f(x)f'(x)dx = f(b)^2 - f(a)^2$$

$$\Rightarrow f(x) = Ci(x) \Rightarrow f'(x) = \frac{\cos x}{x}$$

$$\Rightarrow \int_a^b \left(Ci(x)^2 + \left(\frac{\cos x}{x}\right)^2\right)dx \geq f(b)^2 - f(a)^2$$

2397. Find:

$$\Omega = \lim_{n\to\infty}\left(\log n + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Proposed by Khaled Abd Imouti – Syria

Solution by Amin Hajiyev – Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(\log(n) + \lim_{n\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right) = \lim_{n\to 0}(\log(n) + f(n))$$

$$f(n) = \lim_{n\to 0}\frac{1+x^2-(1+x^2)^{H_n}}{x^2(1+x^2)}\ \{1+x^2 = t\ \ x^2 = t-1\}$$

$$f(n) = \lim_{t\to 1}\frac{t-t^{H_n}}{t^2-t} = \lim_{x\to 1}\frac{\frac{\partial}{\partial t}(t-t^{H_n})}{\frac{\partial}{\partial t}(t^2-t)} = \lim_{t\to 1}\frac{1-H_n t^{H_n-1}}{2t-1} = 1-H_n$$

$$\Omega = \lim_{n\to\infty}(\log(n) - H_n + 1) = \lim_{n\to\infty}\left(\log(n) - \ln(n) - \gamma - \frac{1}{2n} + \xi_n + 1\right) = 1-\gamma$$

$$\left\{0 \leq \xi_n \leq \frac{1}{8n^2}\ \ n\to\infty\ \ \ \xi_n \to 0\right\}$$

2398. Find:

$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^2y^2(y^2+1)(x+1)^2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pham Duc Nam-Vietnam

$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^2y^2(y^2+1)(x+1)^2} = \sum_{x=1}^{\infty}\frac{1}{x^2(x+1)^2}\sum_{y=1}^{\infty}\frac{1}{y^2(y^2+1)} =$$

$$\sum_{x=1}^{\infty}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{2}{x-1} - \frac{2}{x}\right)\sum_{y=1}^{\infty}\left(\frac{1}{y^2} - \frac{1}{y^2+1}\right) =$$

$$\left(\sum_{x=1}^{\infty}\left(\frac{1}{x^2}+\frac{1}{(x+1)^2}\right)+\underbrace{\sum_{x=1}^{\infty}\left(\frac{2}{x+1}-\frac{2}{x}\right)}_{Telescoping\ \ sum}\right)\sum_{y=1}^{\infty}\left(\frac{1}{y^2}-\frac{1}{y^2+1}\right)=$$

$$\left(-1+\frac{\pi^2}{3}-2\right)\left(\frac{\pi^2}{6}-\sum_{y=1}^{\infty}\frac{1}{y^2+1}\right)=\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{1}{2}\sum_{y=-\infty}^{\infty}\frac{1}{y^2+1}-\frac{1}{2}\right)\right)=$$

$$\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{1}{2}\left(-\pi Res\left(\frac{1}{1+z^2}\cot(\pi z), z=\pm i\right)\right)-\frac{1}{2}\right)\right)=$$

$$\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\left(\frac{\pi}{2}\cot(\pi)-\frac{1}{2}\right)\right)=\left(\frac{\pi^2}{3}-3\right)\left(\frac{\pi^2}{6}-\frac{\pi}{2}\cot(\pi)+\frac{1}{2}\right)=$$

$$\frac{1}{18}(\pi^2-9)(\pi^2-3\pi\cot(\pi)+3)$$

2399. Prove that:

$$\sum_{n=1}^{\infty}\frac{6^nH_n^{(2)}+3^nnH_n^{(2)}+2^nn^2H_n^{(3)}}{(-1)^nn6^n}=\frac{\pi^2}{12}\ln(2)-\zeta(3)+\frac{2}{3}Li_2\left(-\frac{1}{2}\right)$$
$$+\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sigma=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{n}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{2^n}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(3)}}{3^n}n=\sigma_1+\sigma_2+\sigma_3$$

Notes:

$$\left\{\sum_{n=1}^{\infty}(-1)^nH_n^{(q)}x^n=\frac{Li_q(-x)}{1+x}\right\};\left\{\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^nH_n^{(q)}x^n=\frac{d}{dx}\frac{Li_q(-x)}{1+x}\right\}$$

$$\sigma_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{n}=\sum_{n=1}^{\infty}(-1)^nH_n^{(2)}\int_0^1x^{n-1}dx=\int_0^1\frac{1}{x}\sum_{n=1}^{\infty}(-1)^nH_n^{(2)}x^ndx=\int_0^1\frac{Li_2(-x)}{x(1+x)}dx$$
$$=$$

$$=\int_0^1\frac{Li_2(-x)}{x}dx-\int_0^1\frac{Li_2(-x)}{1+x}dx \underset{IBP}{}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^{n-1}dx-[ln(1+x)Li_2(-x)]_0^1-\int_0^1\frac{\frac{ln^2(1+x)}{x}}{\{\frac{1}{1+x}=x\}}dx=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-Li_2(-1)ln(2)-\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x}dx-\int_{\frac{1}{2}}^1\frac{ln^2(x)}{1-x}dx$$

$$=-\eta(3)+\frac{\zeta(2)ln(2)}{2}-[\frac{ln^3(x)}{3}]_{\frac{1}{2}}^1-\int_0^1\frac{ln^2(x)}{1-x}dx+\int_0^{\frac{1}{2}}\frac{ln^2(x)}{1-x}dx=$$

$$=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)-\frac{ln^3\left(\frac{1}{2}\right)}{3}-\sum_{n=0}^{\infty}\int_0^1 x^n ln^2(x)dx+\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}}x^n ln^2(x)dx=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)$$

$$+\frac{ln^3(2)}{3}-2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}+\sum_{n=0}^{\infty}([\frac{x^{n+1}ln^2(x)}{n+1}]_0^{\frac{1}{2}}$$

$$-\frac{2}{n+1}\int_0^{\frac{1}{2}}x^n ln(x)dx=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)+\frac{ln^3(2)}{3}-2\zeta(3)-\frac{ln^3(2)}{3}+\frac{7\zeta(3)}{4}$$

$$=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)-\frac{\zeta(3)}{4}=\frac{\pi^2\ln(2)}{12}-\zeta(3)$$

$$\sigma_2=\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{2^n}=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n H_n^{(2)}=\frac{Li_2\left(-\frac{1}{2}\right)}{1+\frac{1}{2}}=\frac{2Li_2\left(-\frac{1}{2}\right)}{3}$$

$$\sigma_3=\sum_{n=1}^{\infty}\frac{(-1)^n nH_n^{(3)}}{3^n}$$

$$\left\{\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^n x^n H_n^{(3)}=\frac{d}{dx}\frac{Li_3(-x)}{1+x};\ \sum_{n=1}^{\infty}(-1)^n nx^{n-1}H_n^{(3)}=\frac{(x+1)Li_2(-x)-xLi_3(-x)}{x(x+1)^2}\right\}$$

$$\sigma_3=\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{3}\right)^n H_n^{(3)}=\frac{\left(1+\frac{1}{3}\right)Li_2\left(-\frac{1}{3}\right)-\frac{1}{3}Li_3\left(-\frac{1}{3}\right)}{\left(1+\frac{1}{3}\right)^2}=\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$

$$\sum_{n=1}^{\infty}\frac{6^n H_n^{(2)}+3^n nH_n^{(2)}+2^n n^2 H_n^{(3)}}{(-1)^n 6^n n}=\sigma_1+\sigma_2+\sigma_3=\frac{\pi^2 ln(2)}{12}-\zeta(3)+\frac{2}{3}Li_2\left(-\frac{1}{2}\right)$$

$$+\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$

2400. Find:

$$\sum_{n=1}^{\infty}\frac{H_n-H_{\frac{n}{2}}}{n}\binom{2n}{n}\left(-\frac{1}{4}\right)^n$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by proposer

$$\sum_{n=1}^{\infty}\frac{H_n-H_{\frac{n}{2}}}{n}\binom{2n}{n}\left(-\frac{1}{4}\right)^n=\sum_{n=1}^{\infty}\frac{\Gamma(2n+1)}{n!\,n2^{2n}\Gamma(n+1)}(-1)^n\int_0^1\left(\frac{1-x^n}{1-x}-\frac{1-x^{\frac{n}{2}}}{1-x}\right)dx$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\,n}(-1)^n\int_0^1\frac{x^{\frac{n}{2}}-x^n}{1-x}dx$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n\int_0^1\left(\frac{1}{2}x^{\frac{n}{2}-1}-x^{n-1}\right)\ln(1-x)\,dx$$

$$=\int_0^1\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n x^{\frac{n}{2}-1}-\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n x^{n-1}\right)\ln(1-x)\,dx$$

$$=\int_0^1\left(\frac{1}{2}\frac{1}{x\sqrt{1+\sqrt{x}}}-\frac{1}{x\sqrt{1+x}}+\frac{1}{2x}\right)\ln(1-x)\,dx$$

$$=\frac{1}{2}\int_0^1\frac{1}{x\sqrt{1+\sqrt{x}}}\ln(1-x)\,dx-\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\frac{\pi^2}{12}=$$

$$=\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\frac{\pi^2}{12}=$$

$$=\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1+x)\,dx-\frac{\pi^2}{12}$$

$$=4\int_1^{\sqrt{2}}\frac{1}{(u^2-1)}\ln u\,du-\frac{\pi^2}{12}=-4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{1-t^2}\ln t\,dt-\frac{\pi^2}{12}=$$

$$=-2\int_0^{\frac{\sqrt{2}-1}{\sqrt{2}+1}}\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)dx-\frac{\pi^2}{12}=2Li_2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)-2Li_2\left(-\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)-\frac{\pi^2}{12}$$

Solution 2 by Bui Hong Suc-Vietnam

By Taylor series:

$$\frac{1}{\sqrt{1+x}}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}(-x)^n=1+\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}(-x)^n\to\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n x^n=\frac{1-\sqrt{1+x}}{\sqrt{1+x}}$$

Multiply through by $\frac{\ln(1+x)}{x}$ then integrate using $\int_0^1 x^{n-1}\ln(1+x)\,dx=\frac{H_n-H_{\frac{n}{2}}}{n}$,

$$\therefore\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n\frac{H_n-H_{\frac{n}{2}}}{n}=\int_0^1\frac{\left(1-\sqrt{1+x}\right)\ln(1+x)}{x\sqrt{1+x}}dx=$$

$$=\int_0^1\frac{\left(1-\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)\ln(1+x)}{x\left(1+\sqrt{1+x}\right)\sqrt{1+x}}dx$$

$$=2\int_0^1\frac{\ln\left(\frac{1}{\sqrt{1+x}}\right)}{\left(1+\sqrt{1+x}\right)\sqrt{1+x}}dx\overset{\frac{1}{v^2}=1+x}{=}4\int_{\frac{1}{\sqrt{2}}}^1\frac{\ln(v)}{\left(1+\frac{1}{b}\right)\frac{1}{v}}\frac{dv}{v^3}=$$

$$=4\int_{\frac{1}{\sqrt{2}}}^1\frac{\ln(v)}{(v+1)v}=4\int_{\frac{1}{\sqrt{2}}}^1\frac{\ln(v)}{v}dv-4\int_{\frac{1}{\sqrt{2}}}^1\frac{\ln(v)}{v+1}dv$$

$$=2\ln^2(v)|_{\frac{1}{\sqrt{2}}}^1-4\ln(v)\ln(1+v)\,|_{\frac{1}{\sqrt{2}}}^1+4\int_{\frac{1}{\sqrt{2}}}^1\frac{\ln\left(1-(-v)\right)}{-v}d(-v)=$$

$$=-2\ln^2\sqrt{2}-4\ln\left(\sqrt{2}\right)\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)-4Li_2(-v)|_{\frac{1}{\sqrt{2}}}^1$$

$$=4Li_2\left(-\frac{1}{\sqrt{2}}\right)-4Li_2(-1)+2\ln^2\sqrt{2}-4\ln\left(\sqrt{2}\right)\ln\left(1+\sqrt{2}\right)=$$

$$=4Li_2\left(-\frac{1}{\sqrt{2}}\right)+\frac{\pi^2}{3}+2\ln^2\sqrt{2}-2\ln(2)\ln\left(1+\sqrt{2}\right)$$

Hence:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n\frac{H_n-H_{\frac{n}{2}}}{n}=4Li_2\left(-\frac{1}{\sqrt{2}}\right)+\frac{\pi^2}{3}+2\ln^2\sqrt{2}-2\ln(2)\ln\left(1+\sqrt{2}\right)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru