A SIMPLE PROOF FOR DURELL'S INEQUALITY

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Abstract. In this paper we will give a simple proof for Durell's inequality in any triangle ABC and a few connections with Doucet's, Euler's and Mitrinovic's inequalities.

DURELL'S INEQUALITY

In any triangle ABC the following inequality holds:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1 \tag{1}$$

Proof.

Lemma 1.

If $x,y,z\in\mathbb{R}$:

$$3(xy+yz+zx)\leq (x+y+z)^2 \tag{2}$$

Solution.

$$3xy+3yz+3z\leq x^2+y^2+z^2+2xy+2yz+2zx$$
$$x^2+y^2+z^2\geq xy+yz+zx$$
$$2x^2+2y^2+2z^2\geq 2xy+2yz+2zx$$
$$x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0$$
$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

□

Lemma 2.

If r_a,r_b,r_c are exradies in ΔABC then:

$$r_ar_b+r_br_c+r_cr_a=s^2 \tag{3}$$

Solution.

$$r_ar_b+r_br_c+r_cr_a=\sum_{cyc}r_ar_b=\sum_{cyc}\frac{F}{s-a}\cdot\frac{F}{s-b}=$$
$$=\sum_{cyc}\frac{F^2}{(s-a)(s-b)}=\sum_{cyc}\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}=$$
$$=s\sum_{cyc}(s-c)=s\left(3s-\sum_{cyc}c\right)=s(3s-2s)=s^2$$

□

Lemma 3.

If r_a,r_b,r_c are exradies in ΔABC then:

$$r_a+r_b+r_c=r+4R \tag{4}$$

Solution.

$$r_a+r_b+r_c=\sum_{cyc} r_a=\sum_{cyc}\frac{F}{s-a}=F\sum_{cyc}\frac{1}{s-a}=$$
$$=\frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}(s-b)(s-c)=$$
$$=\frac{Fs}{s(s-a)(s-b)(s-c)}\sum_{cyc}(s^2-s(b+c)+bc)=$$
$$=\frac{Fs}{F^2}\sum_{cyc}(s^2-s(2s-a)+bc)=$$
$$=\frac{s}{F}\left(3s^2-6s^2+s\sum_{cyc}a+\sum_{cyc}bc\right)=$$
$$=\frac{s}{rs}(-3s^2+2s^2+s^2+r^2+4Rr)=$$
$$=\frac{1}{r}(r^2+4Rr)=r+4R$$

□

Lemma 4. (Doucet's inequality)
In ΔABC the following relationship holds:

$$s\sqrt{3}\le r+4R \tag{5}$$

Solution.
We replace in $(2): x=r_a; y=r_b; z=r_c$

$$3(r_ar_b+r_br_c+r_cr_a)\le(r_a+r_b+r_c)^2$$

By $(3);(4)$:

$$3s^2\le(r+4R)^2$$
$$s\sqrt{3}\le r+4R$$

□

Back to the main result:
Durell's inequality (1) can be written:

$$\sum_{cyc}\tan^2\frac{A}{2}\ge 1$$
$$\sum_{cyc}\frac{(s-b)(s-c)}{s(s-a)}\ge 1$$
$$\sum_{cyc}(s-b)^2(s-c)^2\ge s(s-a)(s-b)(s-c)=F^2$$

By C-B-S inequality:

$$\sum_{cyc}(s-b)^2(s-c)^2\ge\frac{1}{3}\left(\sum_{cyc}(s-b)(s-c)\right)^2=$$
$$=\frac{1}{3}\left(\sum_{cyc}(s^2-s(b+c)+bc)\right)^2=$$

$$= \frac{1}{3}\left(3s^2 - s\sum_{cyc}(2s-a) + \sum_{cyc} bc\right)^2 =$$
$$= \frac{1}{3}\left(3s^2 - 6s^2 + s\sum_{cyc} a + s^2 + r^2 + 4Rr\right)^2 =$$
$$= \frac{1}{3}(-3s^2 + 2s^2 + s^2 + r^2 + 4Rr)^2 =$$
$$= \frac{1}{3}(r^2 + 4Rr)^2$$

□

Remains to prove that:

$$\frac{1}{3}(r^2 + 4Rr)^2 \geq F^2$$
$$r^2(r+4R)^2 \geq 3r^2s^2$$
$$3s^2 \leq (r+4R)^2$$
$$s\sqrt{3} \leq r + 4R$$

which is (5) - Doucet's inequality.
Using Euler's inequality:

$$r \leq \frac{R}{2}$$

we can obtain by (5):

$$s\sqrt{3} \leq r + 4R \leq \frac{R}{2} + 4R = \frac{9R}{2}$$

$$s \leq \frac{9R}{2\sqrt{3}} = \frac{3\sqrt{3}R}{2} \tag{6}$$

which is Mitrinovič inequality.
Another proof for (6) it is based on the concavity of the function:

$$f : (0, \pi) \to \mathbb{R}; f(x) = \sin x$$
$$f'(x) = \cos x; f''(x) = -\sin x < 0$$

By Jensen's inequality:

$$f(A) + f(B) + f(C) \leq 3f\left(\frac{A+B+C}{3}\right)$$
$$\sin A + \sin B + \sin C \leq 3\sin\frac{\pi}{3}$$
$$2R\sin A + 2R\sin B + 2R\sin C \leq 6R \cdot \frac{\sqrt{3}}{2}$$
$$a + b + c \leq 3\sqrt{3}R$$
$$\frac{a+b+c}{2} \leq \frac{3\sqrt{3}}{2}R$$
$$s \leq \frac{3\sqrt{3}}{2}R$$

Observation 1:
In ΔABC (not right angled), the following relationship holds:

$$\cot^2 A + \cot^2 B + \cot^2 C \geq 1$$

Proof.
Let's consider the triangle with angles:

$$\pi - 2A; \pi - 2B; \pi - 2C$$

Let's observe that:

$$(\pi - 2A) + (\pi - 2B) + (\pi - 2C) = 3\pi - 2(A + B + C) = 3\pi - 2\pi = \pi$$

By (1):

$$\tan^2\left(\frac{\pi - 2A}{2}\right) + \tan^2\left(\frac{\pi - 2B}{2}\right) + \tan^2\left(\frac{\pi - 2C}{2}\right) \geq 1$$
$$\tan^2\left(\frac{\pi}{2} - A\right) + \tan^2\left(\frac{\pi}{2} - B\right) + \tan^2\left(\frac{\pi}{2} - C\right) \geq 1$$
$$\cot^2 A + \cot^2 B + \cot^2 C \geq 1$$

□

Observation 2:
In ΔABC the following relationship holds:

$$\tan^2 \frac{5A}{2} + \tan^2 \frac{5B}{2} + \tan^2 \frac{5C}{2} \geq 1$$

Proof.
Let's consider the triangle with angles:

$$2\pi - 5A; 2\pi - 5B; 2\pi - 5C$$

Let's observe that:

$$(2\pi - 5A) + (2\pi - 5B) + (2\pi - 5C) =$$
$$= 6\pi - 5(A + B + C) = 6\pi - 5\pi = \pi$$

By (1):

$$\tan^2\left(\frac{2\pi - 5A}{2}\right) + \tan^2\left(\frac{2\pi - 5B}{2}\right) + \tan^2\left(\frac{2\pi - 5C}{2}\right) \geq 1$$
$$\tan^2\left(\pi - \frac{5A}{2}\right) + \tan^2\left(\pi - \frac{5B}{2}\right) + \tan^2\left(\pi - \frac{5C}{2}\right) \geq 1$$
$$\tan^2 \frac{5A}{2} + \tan^2 \frac{5B}{2} + \tan^2 \frac{5C}{2} \geq 1$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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