IONESCU - WEITZENBÖCK'S TYPE INEQUALITIES WITH FIBONACCI NUMBERS

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Abstract. In this paper we present some inequalities with Fibonacci numbers related to Ionescu - Weitzenböck's inequality

Let m positive real number and n be positive integer number. If ABC, is a triangle with area S and usual notations then we have that:

$$(1)\quad \frac{a^{2(m+1)}}{(F_n m_a^2 + F_{n+1} m_b^2)^m} + \frac{b^{2(m+1)}}{(F_n m_b^2 + F_{n+1} m_c^2)} + \frac{c^{2(m+1)}}{(F_n m_c^2 + F_{n+1} m_a^2)} \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m} S$$

$$(2)\quad \frac{m_a^{2(m+1)}}{(F_n b^2 + F_{n+1} c^2)^m} + \frac{m_b^{2(m+1)}}{(F_n c^2 + F_{n+1} a^2)} + \frac{m_c^{2(m+1)}}{(F_n a^2 + F_{n+1} b^2)} \geq \frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m} S$$

$$(3)\quad \frac{a^{2(m+1)}}{(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2)^m} + \frac{b^{2(m+1)}}{(F_n m_b^2 + F_{n+1} m_c^2 + F_{n+2} m_a^2)} + $$
$$+ \frac{c^{2(m+1)}}{(F_n m_c^2 + F_{n+1} m_a^2 + F_{n+2} m_b^2)} \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m} S$$

$$(4)\quad \frac{m_a^{2(m+1)}}{(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2)^m} + \frac{m_b^{2(m+1)}}{(F_n b^2 + F_{n+1} c^2 + F_{n+2} a^2)} + $$
$$+ \frac{m_c^{2(m+1)}}{(F_n c^2 + F_{n+1} a^2 + F_{n+2} b^2)} \geq \frac{3^{m+1}\sqrt{3}}{8^m F_{n+2}^m} S$$

Proof.
We use Radon's inequality, the well-known formula

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2),$$

and Ionescu-Weitzenböck's inequality, i.e.

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} S$$

□

Proof of (1).
We have:

$$W_n = \sum \frac{a^{2(m+1)}}{(F_n m_a^2 + F_{n+1} m_b^2)^m} = \sum \frac{(a^2)^{m+1}}{(F_n m_a^2 + F_{n+1} m_b^2)^m},$$

and by J. Radon's inequality we deduce that:

$$W_n \geq \frac{(a^2 + b^2 + c^2)^{m+1}}{(\sum (F_n m_a^2 + F_{n+1} m_b^2))^m} = \frac{(a^2 + b^2 + c^2)^{m+1}}{(F_n + F_{n+1})^m (m_a^2 + m_b^2 + m_c^2)^m} =$$

$$= \frac{(a^2+b^2+c^2)^{m+1}}{F_{n+2}^m(m_a^2+m_b^2+m_c^2)^m}$$

But, $m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$, so: $W_n \geq \frac{4^m}{3^m F_{n+2}^m}(a^2+b^2+c^2)$.

By Ionescu - Weitzenböck's inequality, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$, we deduce that:

$$W_n \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m}S, \text{ so (1) is proved.}$$

□

Proof of (2).
We have:

$$Y_n = \sum \frac{m_a^{2(m+1)}}{(F_n b^2 + F_{n+1}c^2)^m} = \sum \frac{(m_a^2)^{m+1}}{(F_n b^2 + F_{n+1}c^2)^m},$$

and by J. Radon's inequality we deduce that:

$$Y_n \geq \frac{(m_a^2+m_b^2+m_c^2)^{m+1}}{(\sum(F_n b^2+F_{n+1}c^2))^m} = \frac{(m_a^2+m_b^2+m_c^2)^{m+1}}{(F_n+F_{n+1})^m(a^2+b^2+c^2)^m} =$$

$$= \frac{(m_a^2+m_b^2+m_c^2)^{m+1}}{F_{n+2}^m(a^2+b^2+c^2)^m}.$$

Since, $m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$, we have:

$$Y_n \geq \frac{3^{m+1}}{4^{m+1}F_{n+2}^m}(a^2+b^2+c^2).$$

By Ionescu - Weitzenböck's inequality, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$, we deduce that:

$$Y_n \geq \frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m}S, \text{ so (2) is proved.}$$

□

Proof of (3).
We have:

$$Z_n = \sum \frac{a^{2(m+1)}}{(F_n m_a^2 + F_{n+1}m_b^2 + F_{n+2}m_c^2)^m} = \sum \frac{(a^2)^{m+1}}{(F_n m_a^2 + F_{n+1}m_b^2 + F_{n+2}m_c^2)^m},$$

and by J. Radon's inequality we deduce that:

$$Z_n \geq \frac{(a^2+b^2+c^2)^{m+1}}{(\sum(F_n m_a^2+F_{n+1}m_b^2+F_{n+2}m_c^2))^m} = \frac{(a^2+b^2+c^2)^{m+1}}{(F_n+F_{n+1}+F_{n+2})^m(m_a^2+m_b^2+m_c^2)^m} =$$

$$= \frac{(a^2+b^2+c^2)^{m+1}}{2^m F_{n+2}^m(m_a^2+m_b^2+m_c^2)^m}.$$

Since, $m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$, we have:

$$Z_n \geq \frac{2^m}{3^m F_{n+2}^m}(a^2+b^2+c^2).$$

By Ionescu - Weitzenböck's inequality, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$, we deduce that:

$$Z_n \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m}S, \text{ so (3) is proved.}$$

□

Proof of (4).
We have:

$$X_n = \sum \frac{m_a^{2(m+1)}}{(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2)^m} = \sum \frac{(m_a^2)^{m+1}}{(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2)^m},$$

and by J. Radon's inequality we deduce that:

$$X_n \geq \frac{(m_a^2 + m_b^2 + m_c^2)^{m+1}}{(\sum (F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2))^m} = \frac{(m_a^2 + m_b^2 + m_c^2)^{m+1}}{(F_n + F_{n+1} + F_{n+2})^m (a^2 + b^2 + c^2)^m} =$$

$$= \frac{(m_a^2 + m_b^2 + m_c^2)^{m+1}}{2^m F_{n+2}^m (a^2 + b^2 + c^2)^m}.$$

We know that: $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$, so:

$$X_n \geq \frac{3^{m+1}}{2^{3m+2} F_{n+2}^m} (a^2 + b^2 + c^2).$$

By Ionescu - Weitzenböck's inequality, i.e. $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$, we obtain that:

$$X_n \geq \frac{3^{m+1}\sqrt{3}}{2^{3m} F_{n+2}^m} S, \text{ and the proof is complete.}$$

□

References

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