LALESCU AND EULER - MASCHERONI TYPE NEW LIMITS

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Abstract. In this paper we present new limits with sequences.

$$\text{I. } \left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}=\frac{2\pi}{e}.$$

Proof.

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e,\ \lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}.$$

Denoting $u_n=\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$: $\lim\limits_{n\to\infty}u_n=1,\ \lim\limits_{n\to\infty}\dfrac{u_n-1}{\ln u_n}=1,\ \lim\limits_{n\to\infty}u_n^n=e.$

We denote $x_n=\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\dfrac{\pi}{\sqrt[n]{n!}}$:

$$x_n=\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}=$$

$$=\sqrt[n]{n!}\cdot(u_n-1)\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\sqrt[n]{n!}\cdot\sin\frac{\pi}{\sqrt[n]{n!}}=$$

$$=\frac{\sqrt[n]{n!}}{n}\cdot n\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n\cdot\frac{(2n-1)!!}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\pi\frac{\sin\frac{\pi}{\sqrt[n]{n!}}}{\frac{\pi}{\sqrt[n]{n!}}}=$$

$$=\pi\cdot\frac{\sqrt[n]{n!}}{n!}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\cdot\frac{(2n-1)!!}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{\sin\frac{\pi}{\sqrt[n]{n!}}}{\frac{\pi}{\sqrt[n]{n!}}}$$

$$\text{Hence, }\lim_{n\to\infty}x_n=\pi\cdot\frac{1}{e}\cdot 1\cdot\ln e\cdot\frac{2}{e}\cdot e\cdot 1=\frac{2\pi}{e}$$

□

II. $\lim\limits_{n\to\infty}\sqrt{n}(\sqrt{a_{n+1}}-\sqrt{a_n})=\dfrac{\sqrt{a}}{2}$, where $(a_n)_{n\geq 1}$ is a positive sequence such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a, a>0.$$

Proof.

$\lim\limits_{n\to\infty}\dfrac{a_n}{n}\overset{\text{C-S}}{=}\lim\limits_{n\to\infty}\dfrac{a_{n+1}-a_n}{(n+1)-n}=a$ denoting $x_n=\sqrt{n}(\sqrt{a_{n+1}}-\sqrt{a_n})=\sqrt{n}\cdot\dfrac{a_{n+1}-a_n}{\sqrt{a_{n+1}}-\sqrt{a_n}}=$

$$=(a_{n+1}-a_n)\cdot\frac{1}{\sqrt{\frac{a_{n+1}}{n+1}}\cdot\sqrt{\frac{n+1}{n}}+\sqrt{\frac{a_n}{n}}};\text{ so}$$

$$\lim_{n\to\infty}\sqrt{n}(\sqrt{a_{n+1}}-\sqrt{a_n})=\lim_{n\to\infty}x_n=a\cdot\frac{1}{\sqrt{a}\cdot 1+\sqrt{a}}=\frac{\sqrt{a}}{2}.$$

□

III. $\lim_{n\to\infty} e^{-H_n} \sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = 2\cdot e^{-(\gamma+1)}$, where

$$(2k-1)!! = 1\cdot 3\cdot\ldots\cdot(2k-1), \forall k\in\mathbb{N}^* \text{ and } H_n = \sum_{k=1}^{n}\frac{1}{k}.$$

Proof.

$$n\cdot e^{-H_n} = e^{\ln n}e^{-H_n} = e^{\ln n - H_n} = e^{-\gamma_n}, \text{ where}$$
$$\lim_{n\to\infty}\gamma_n = \gamma \text{ is Euler - Mascheroni constant.}$$

$$x_n = e^{-H_n}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = ne^{-H_n}\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n} =$$
$$= e^{-\gamma_n}\cdot\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}};\ \lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}} \overset{\text{C-D}}{=}$$
$$= \lim_{n\to\mathcal{R}}\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n+1]{(2n+1)!!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = \lim_{n\to\mathcal{R}}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} =$$
$$= \lim_{n\to\mathcal{R}}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\mathcal{R}}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{C-D}}{=}$$
$$\overset{\text{C-D}}{=} \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} = \lim_{n\to\mathcal{R}}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}.$$
$$\text{Hence, } \lim_{n\to\mathcal{R}} x_n = e^{-\gamma}\cdot\frac{2}{e} = 2\cdot e^{-(\gamma+1)}.$$

□

IV. $\lim_{n\to\infty}\left(\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b} - \sqrt[n]{(n!)^a((2n-1)!!)^b}\right) =$

$$= \begin{cases} 0, \text{ if } a+b<1 \\ \frac{2^b}{e}, \text{ if } a+b=1, \text{ where } a,b\in\mathbb{R} \\ \infty, \text{ if } a+b>1 \end{cases}$$

Proof.

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e,\ \lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \frac{2}{e}.$$

Denoting $u_n = \dfrac{\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b}}{\sqrt[n]{(n!)^a((2n-1)!!)^b}}$: $\lim_{n\to\infty}u_n = 1$, $\lim_{n\to\infty}\dfrac{u_n-1}{\ln u_n} = 1$;

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\mathcal{T}}\frac{((n+1)!)^a}{(n!)^a}\cdot\frac{((2n+1)!!)^b}{((2n-1)!!)^b}\cdot\frac{1}{\sqrt[n+1]{((n+1)!)^a(((2n+1)!!)^b}} =$$
$$= \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^a\left(\frac{2n+1}{n+1}\right)^b\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)^b = e^a2^b\left(\frac{e}{2}\right)^b = e^{a+b}$$

We denote $x_n = \left(\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b} - \sqrt[n]{(n!)^a((2n-1)!!)^b}\right) =$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^a\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^b\cdot n^{a+b-1}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$$

Hence, $\lim\limits_{n\to\infty} x_n = \frac{1}{e^a} \cdot \frac{2^b}{e^b} \cdot 1 \cdot \ln e^{a+b} \cdot \lim\limits_{n\to\mathcal{R}} n^{a+b-1} =$

$$= \begin{cases} 0, \text{ if } a+b<1 \\ \frac{2^b}{e}, \text{ if } a+b=1 \\ \infty, \text{ if } a+b>1 \end{cases}$$

□

References

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