RMM - Cyclic Inequalities Marathon 1401 - 1500

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1401. Let $a, b, c > 0$. Prove that:

$$a^b + b^c + c^a > 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

If at least one of the numbers a, b and c is greater than or equal to 1, inequality occurs.

Assume now that $a, b, c < 1$. By Bernoulli's inequality, we have

$$a^b = \frac{a}{[1+(a-1)]^{1-b}} \geq \frac{a}{1+(a-1)(1-b)} = \frac{a}{a+b-ab} > \frac{a}{a+b+c} \text{ (and analogs)}$$

Therefore

$$a^b + b^c + c^a > \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1,$$

which completes the proof.

1402. If $a, b, c, x, y, z \in \mathbb{R}$, then :

$$\frac{a^2+b^2}{x^2+y^2} + \frac{b^2+c^2}{y^2+z^2} + \frac{c^2+a^2}{z^2+x^2} \geq \left(\frac{a+b+c}{x^2+y^2+z^2}\right) . \sqrt{3(a^2+b^2+c^2)}$$

When does equality hold ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2+b^2}{x^2+y^2} + \frac{b^2+c^2}{y^2+z^2} + \frac{c^2+a^2}{z^2+x^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}\sqrt{b^2+c^2}\right)^2}{2\sum_{cyc}x^2}$$

$$= \frac{\sum_{cyc}(b^2+c^2) + 2\sum_{cyc}\left(\sqrt{b^2+c^2}.\sqrt{c^2+a^2}\right)}{2\sum_{cyc}x^2}$$

$$= \frac{\sum_{cyc}a^2 + \sqrt{\sum_{cyc}(b^2+c^2)(c^2+a^2) + 2\sum_{cyc}\left((c^2+a^2).\sqrt{b^2+c^2}.\sqrt{a^2+b^2}\right)}}{\sum_{cyc}x^2}$$

$$\geq \frac{\sum_{cyc}a^2 + \sqrt{\sum_{cyc}a^4 + 3\sum_{cyc}a^2b^2 + 2\sum_{cyc}\left((c^2+a^2)\left(\frac{b+c}{\sqrt{2}}\right)\left(\frac{a+b}{\sqrt{2}}\right)\right)}}{\sum_{cyc}x^2}$$

$$= \frac{\sum_{cyc}a^2 + \sqrt{\sum_{cyc}a^4 + 3\sum_{cyc}a^2b^2 + \sum_{cyc}\left((c^2+a^2)(\sum_{cyc}ab + b^2)\right)}}{\sum_{cyc}x^2}$$

$$= \frac{\sum_{cyc}a^2 + \sqrt{\sum_{cyc}a^4 + 3\sum_{cyc}a^2b^2 + 2(\sum_{cyc}ab)(\sum_{cyc}a^2) + 2\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+3\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$\geq\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$\Rightarrow\boxed{\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\overset{(*)}{\geq}\frac{m+|m+n|}{\sum_{cyc}x^2}}\left(m=\sum_{cyc}a^2\text{ and }n=\sum_{cyc}ab\right)$$

$$\text{Again,}\boxed{\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A}=\frac{\left(\sum_{cyc}a\right)\left(\sqrt{3\sum_{cyc}a^2}\right)}{2\sum_{cyc}x^2}$$

$$+\frac{3\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2-2\left(\sum_{cyc}a\right)\left(\sqrt{3\sum_{cyc}a^2}\right)}{2\sum_{cyc}x^2}$$

$$\left(\text{where : }A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}\right)=\frac{4\sum_{cyc}a^2+2\sum_{cyc}ab}{2\sum_{cyc}x^2}$$

$$\boxed{=\frac{2m+n}{\sum_{cyc}x^2}\leq\frac{m+|m+n|}{\sum_{cyc}x^2}}(\because|t|\geq t)\overset{\text{via }(*)}{\leq}\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}$$

$$\therefore\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\geq\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}$$

$$+\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

$$\geq\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)},\text{ ''='' iff }(a=b=c=0)$$

$$\text{or }(a=b=c\neq0\text{ and }|x|=|y|=|z|)\text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\geq\frac{\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2}{(x^2+y^2)+(y^2+z^2)+(z^2+x^2)}$$

$$=\frac{a^2+b^2+c^2+\sqrt{(a^2+b^2)(a^2+c^2)}+\sqrt{(b^2+c^2)(b^2+a^2)}+\sqrt{(c^2+a^2)(c^2+b^2)}}{x^2+y^2+z^2}$$

$$\geq \frac{a^2+b^2+c^2+(a^2+bc)+(b^2+ca)+(c^2+ab)}{x^2+y^2+z^2} = \frac{(a+b+c)^2+3(a^2+b^2+c^2)}{x^2+y^2+z^2}$$

$$\overset{AM-GM}{\geq} \frac{2(a+b+c)\sqrt{3(a^2+b^2+c^2)}}{2(x^2+y^2+z^2)} = \left(\frac{a+b+c}{x^2+y^2+z^2}\right)\sqrt{3(a^2+b^2+c^2)}.$$

as desired. Equality holds iff $(a=b=c=0)$ or $(a=b=c\neq 0$ and $x^2=y^2=z^2)$.

1403. If $a,b,c,x,y,z\in\mathbb{R}$, then :

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\geq\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A,$$

$$\text{where : } A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

When does equality holds ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\sqrt{b^2+c^2}\right)^2}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}(b^2+c^2)+2\sum_{cyc}\left(\sqrt{b^2+c^2}.\sqrt{c^2+a^2}\right)}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}(b^2+c^2)(c^2+a^2)+2\sum_{cyc}\left((c^2+a^2).\sqrt{b^2+c^2}.\sqrt{a^2+b^2}\right)}}{\sum_{cyc}x^2}$$

$$\geq\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2\sum_{cyc}\left((c^2+a^2)\left(\frac{b+c}{\sqrt2}\right)\left(\frac{a+b}{\sqrt2}\right)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+\sum_{cyc}\left((c^2+a^2)(\sum_{cyc}ab+b^2)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2(\sum_{cyc}ab)(\sum_{cyc}a^2)+2\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+3\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$\geq\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$\Rightarrow\boxed{\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\overset{(*)}{\geq}\frac{m+|m+n|}{\sum_{cyc}x^2}}\left(m=\sum_{cyc}a^2\ and\ n=\sum_{cyc}ab\right)$$

$$\text{Again},\boxed{\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A}=\frac{\left(\sum_{cyc}a\right)\left(\sqrt{3\sum_{cyc}a^2}\right)}{2\sum_{cyc}x^2}$$

$$+\frac{3\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2-2\left(\sum_{cyc}a\right)\left(\sqrt{3\sum_{cyc}a^2}\right)}{2\sum_{cyc}x^2}$$

$$=\frac{4\sum_{cyc}a^2+2\sum_{cyc}ab}{2\sum_{cyc}x^2}\boxed{=\frac{2m+n}{\sum_{cyc}x^2}\leq\frac{m+|m+n|}{\sum_{cyc}x^2}}$$

$$(\because |t|\geq t)\overset{via\ (*)}{\leq}\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\therefore\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}$$

$$\geq\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A,\text{where}:$$

$$A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)},$$

$$''=''\ \text{iff}\ (a=b=c=0)\ \text{or}\ (a=b=c\neq 0\ and\ |x|=|y|=|z|)\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\geq\frac{\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2}{(x^2+y^2)+(y^2+z^2)+(z^2+x^2)}$$

$$=\frac{a^2+b^2+c^2+\sqrt{(a^2+b^2)(a^2+c^2)}+\sqrt{(b^2+c^2)(b^2+a^2)}+\sqrt{(c^2+a^2)(c^2+b^2)}}{x^2+y^2+z^2}$$

$$\geq\frac{a^2+b^2+c^2+(a^2+bc)+(b^2+ca)+(c^2+ab)}{x^2+y^2+z^2}$$

$$=\frac{2(a+b+c)\sqrt{3(a^2+b^2+c^2)}+\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

$$=\left(\frac{a+b+c}{x^2+y^2+z^2}\right)\sqrt{3(a^2+b^2+c^2)}+A,$$

as desired. Equality holds iff $(a=b=c=0)$ or $(a=b=c\neq 0$ and $x^2=y^2=z^2)$.

1404. If $a,b,c>0, abc=1$ and $\lambda\geq 1$, then :

$$\sum_{cyc}\frac{1}{a^3+b^3+\lambda}\leq\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{a^3+b^3+\lambda}\leq\sum_{cyc}\frac{1}{ab(a+b)+\lambda}\overset{abc=1}{=}\sum_{cyc}\frac{1}{\frac{a+b}{c}+\lambda}=\sum_{cyc}\frac{1}{x+\lambda}$$

$$\left(x=\frac{a+b}{c},y=\frac{b+c}{a},z=\frac{c+a}{b}\right)=\frac{1}{\lambda}\sum_{cyc}\frac{\lambda+x-x}{x+\lambda}=\frac{3}{\lambda}-\frac{1}{\lambda}\sum_{cyc}\frac{x}{x+\lambda}\overset{?}{\leq}\frac{3}{\lambda+2}$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{x+\lambda}\overset{?}{\geq}\frac{6}{\lambda+2}\Leftrightarrow\frac{\sum_{cyc}\left(x(y+\lambda)(z+\lambda)\right)}{(x+\lambda)(y+\lambda)(z+\lambda)}\overset{?}{\geq}\frac{6}{\lambda+2}$$

$$\Leftrightarrow\frac{3xyz+2\lambda\sum_{cyc}xy+\lambda^2\sum_{cyc}x}{xyz+\lambda^3+\lambda\sum_{cyc}xy+\lambda^2\sum_{cyc}x}\overset{?}{\geq}\frac{6}{\lambda+2}\Leftrightarrow$$

$$3\lambda xyz+2\lambda^2\sum_{cyc}xy+\lambda^3\sum_{cyc}x+6xyz+4\lambda\sum_{cyc}xy+2\lambda^2\sum_{cyc}x$$
$$\overset{?}{\geq}6xyz+6\lambda\sum_{cyc}xy+6\lambda^2\sum_{cyc}x+6\lambda^3$$

$$\Leftrightarrow\boxed{\lambda^3\left(\sum_{cyc}x-6\right)+\lambda^2\left(2\sum_{cyc}xy-4\sum_{cyc}x\right)+\lambda\left(3xyz-2\sum_{cyc}xy\right)\underset{(*)}{\overset{?}{\geq}}0}$$

- $$\sum_{cyc}x=\sum_{cyc}\frac{a+b}{c}\overset{abc=1}{=}\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)=\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$$
$$\rightarrow(1)$$
- $$\sum_{cyc}xy=\sum_{cyc}\frac{(a+b)(b+c)}{ca}\overset{abc=1}{=}\sum_{cyc}\left(b\left(b^2+\sum_{cyc}ab\right)\right)=\sum_{cyc}a^3+$$

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \to (2)$$

$$\bullet\ xyz = \prod_{cyc} \frac{a+b}{c} \overset{abc=1}{=} \prod_{cyc}(a+b) \to (3) \therefore \text{via } (1),(2) \text{ and } (3),(*) \Leftrightarrow$$

$$\lambda^3\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right)$$

$$+2\lambda^2\left(\sum_{cyc} a^3 + \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + 6abc\right)$$

$$+\lambda\left(3\prod_{cyc}(a+b) - 2\sum_{cyc} a^3 - 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right) \geq 0$$

$$\Leftrightarrow \lambda^3\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right)$$

$$+2\lambda^2\left(\sum_{cyc} a^3 + 6abc - \left(3abc + \sum_{cyc} a^2b + \sum_{cyc} ab^2\right)\right)$$

$$+\lambda\left(3\prod_{cyc}(a+b) - 2\sum_{cyc} a^3 - 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right) \geq 0$$

$$\Leftrightarrow \boxed{\begin{gathered}\lambda^3\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) + 2\lambda^2\left(\sum_{cyc} a^3 + 3abc - \left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)\right) \\ +\lambda\left(3\prod_{cyc}(a+b) - 2\sum_{cyc} a^3 - 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right) \overset{(**)}{\geq} 0\end{gathered}}$$

$$\text{Now, } \lambda \geq 1 \Rightarrow \lambda^3, \lambda^2 \geq \lambda \text{ and } \because \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc \overset{A-G}{\geq} 0 \text{ and}$$

$$\sum_{cyc} a^3 + 3abc - \left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \overset{\text{Schur}}{\geq} 0 \therefore \text{LHS of } (**) \geq$$

$$\lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) + 2\lambda\left(\sum_{cyc} a^3 + 3abc - \left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)\right)$$

$$+\lambda\left(3\prod_{cyc}(a+b)-2\sum_{cyc}a^3-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)$$

$$=\lambda\left(3\prod_{cyc}(a+b)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2\left(\begin{array}{c}3abc+3abc-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\\ -9abc\end{array}\right)\right)$$

$$=\lambda\left(\begin{array}{c}3\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+12abc\\ -2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\end{array}\right)=0$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}\frac{1}{a^3+b^3+\lambda}\le\frac{3}{\lambda+2}\ \forall\, a,b,c>0\,\big|\,abc=1$$

$$\text{and }\forall\,\lambda\ge 1,''=''\ \text{ iff }(a=b=c=\lambda=1)\ (QED)$$

1405.

$$\text{If } a,b,c>0,\text{ then}:\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3}{2}\cdot\frac{p}{q}\cdot\frac{1}{A},\text{ where}:$$

$$A=1+\frac{1}{8pq}\sum_{cyc}c(a-b)^2,\ p=a+b+c,\ q=ab+bc+ca$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\text{yielding } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x,b=s-y,c=s-z$$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3)\text{ and we have}: A=1+\frac{1}{8pq}\cdot\sum_{cyc}c(a-b)^2$$

$$=1+\frac{1}{8(\sum_{cyc}a)(\sum_{cyc}ab)}\cdot\sum_{cyc}\left(c(a^2+b^2-2ab)\right)$$

$$=1+\frac{1}{8(\sum_{cyc}a)(\sum_{cyc}ab)}\cdot\left(\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)-6abc\right)$$

$$= 1 + \frac{(\sum_{cyc} a)(\sum_{cyc} ab) - 9abc}{8(\sum_{cyc} a)(\sum_{cyc} ab)} \overset{\text{via (1),(2) and (3)}}{=} 1 + \frac{s(4Rr + r^2) - 9r^2 s}{8s(4Rr + r^2)}$$

$$\Rightarrow A = \frac{9R}{2(4R + r)} \therefore \frac{3}{2} \cdot \frac{p}{q} \cdot \frac{1}{A} - \left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right) \overset{\text{via (1) and (3)}}{=}$$

$$\frac{3}{2} \cdot \frac{s}{4Rr + r^2} \cdot \frac{2(4R + r)}{9R} - \left(\sum_{cyc} \frac{1}{x}\right) = \frac{s}{3Rr} - \frac{s^2 + 4Rr + r^2}{4Rrs} \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2 - 12Rr - 3r^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because s^2 - 12Rr - 3r^2$$

$$= s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{\substack{\text{Gerretsen} \\ \text{and} \\ \text{Euler}}}{\geq} 0 \therefore \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}$$

$$\leq \frac{3}{2} \cdot \frac{p}{q} \cdot \frac{1}{A} \; \forall a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

1406. If $a, b, c > 0$, then :

$$\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 + ab} \geq \frac{\sqrt{3(a^2 + b^2 + c^2)}}{2}$$ When does equality hold ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a, b, c, x, y, z > 0, \frac{a^3}{x(a^2 + bc)} + \frac{b^3}{y(b^2 + ca)} + \frac{c^3}{z(c^2 + ab)}$$

$$= \frac{a^4}{ax(a^2 + bc)} + \frac{b^4}{by(b^2 + ca)} + \frac{c^4}{cz(c^2 + ab)}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{ax(a^2 + bc) + by(b^2 + ca) + cz(c^2 + ab)}$$

$$\overset{\text{CBS}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2} \cdot \sqrt{(a^2 + bc)^2 + (b^2 + ca)^2 + (c^2 + ab)^2}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \frac{a^2 + b^2 + c^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

$$\Leftrightarrow 4(a^2 + b^2 + c^2)^2 \overset{?}{\geq} 3\left((a^2 + bc)^2 + (b^2 + ca)^2 + (c^2 + ab)^2\right)$$

$$\Leftrightarrow 4\sum_{cyc} a^4 + 8\sum_{cyc} a^2b^2 \overset{?}{\geq} 3\sum_{cyc} a^4 + 3\sum_{cyc} a^2b^2 + 6abc\sum_{cyc} a$$

$$\Leftrightarrow \sum_{cyc} a^4 + 5\sum_{cyc} a^2b^2 \overset{?}{\geq} 6abc\sum_{cyc} a \rightarrow \text{true} \because \sum_{cyc} a^4 + 5\sum_{cyc} a^2b^2 \geq 6\sum_{cyc} a^2b^2$$

$$\geq 6abc\sum_{cyc} a \therefore \frac{a^3}{x(a^2 + bc)} + \frac{b^3}{y(b^2 + ca)} + \frac{c^3}{z(c^2 + ab)} \geq \frac{\sqrt{3}}{2} \frac{a^2 + b^2 + c^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

$$\forall a, b, c, x, y, z > 0, '' ='' \text{ iff } (a = b = c \text{ and } x = y = z) \rightarrow (1)$$

Implementing (1) with $x = y = z = 1$, we arrive at : $\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 + ab}$

$$\geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2.1+b^2.1+c^2.1}} \therefore \frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2},$$

$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$

1407. If $a,b,c>0$, then :

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab} \geq \frac{a+b+c}{2} \text{ When does equality hold ?}$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c,x,y,z>0, \frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}$$

$$=\frac{a^4}{ax(a^2+bc)}+\frac{b^4}{by(b^2+ca)}+\frac{c^4}{cz(c^2+ab)}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{ax(a^2+bc)+by(b^2+ca)+cz(c^2+ab)}$$

$$\overset{\text{CBS}}{\geq} \frac{(a^2+b^2+c^2)^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}.\sqrt{(a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2}} \overset{?}{\geq} \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\Leftrightarrow 4(a^2+b^2+c^2)^2 \overset{?}{\geq} 3\left((a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2\right)$$

$$\Leftrightarrow 4\sum_{cyc}a^4+8\sum_{cyc}a^2b^2 \overset{?}{\geq} 3\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+6abc\sum_{cyc}a$$

$$\Leftrightarrow \sum_{cyc}a^4+5\sum_{cyc}a^2b^2 \overset{?}{\geq} 6abc\sum_{cyc}a \to \text{true} \because \sum_{cyc}a^4+5\sum_{cyc}a^2b^2 \geq 6\sum_{cyc}a^2b^2$$

$$\geq 6abc\sum_{cyc}a \therefore \frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)} \geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\forall a,b,c,x,y,z>0, '' ='' \text{ iff } (a=b=c \text{ and } x=y=z) \to (1)$$

$$\text{Implementing (1) with } x=y=z=1, \text{we arrive at : } \frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}$$

$$\geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2.1+b^2.1+c^2.1}} = \frac{\sqrt{3(a^2+b^2+c^2)}}{2} \geq \frac{a+b+c}{2}$$

$$\therefore \frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab} \geq \frac{a+b+c}{2}, '' ='' \text{ iff } a=b=c \text{ (QED)}$$

1408. If $a,b,c,x,y,z>0$, then :

$$\frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)} \geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

When does equality hold ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}$$
$$=\frac{a^4}{ax(a^2+bc)}+\frac{b^4}{by(b^2+ca)}+\frac{c^4}{cz(c^2+ab)}$$
$$\overset{\text{Bergstrom}}{\geq}\frac{(a^2+b^2+c^2)^2}{ax(a^2+bc)+by(b^2+ca)+cz(c^2+ab)}$$
$$\overset{\text{CBS}}{\geq}\frac{(a^2+b^2+c^2)^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}.\sqrt{(a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$
$$\Leftrightarrow 4(a^2+b^2+c^2)^2\overset{?}{\geq}3\left((a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2\right)$$
$$\Leftrightarrow 4\sum_{cyc}a^4+8\sum_{cyc}a^2b^2\overset{?}{\geq}3\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+6abc\sum_{cyc}a$$
$$\Leftrightarrow\sum_{cyc}a^4+5\sum_{cyc}a^2b^2\overset{?}{\geq}6abc\sum_{cyc}a\to\text{true}\because\sum_{cyc}a^4+5\sum_{cyc}a^2b^2\geq 6\sum_{cyc}a^2b^2$$
$$\geq 6abc\sum_{cyc}a\therefore\frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}\geq\frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$
$$\forall\, a,b,c,x,y,z>0,''='' \text{ iff } (a=b=c \text{ and } x=y=z)\ (\text{QED})$$

1409. If $a,b,c>0$, *then it is true the inequality* :

$$\sqrt{\sum_{cyc}(a+b)^2}\geq\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x$
$=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)
yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$$\therefore abc=r^2s \text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$
$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$$

$$\text{Now, }\sqrt{\sum_{cyc}(a+b)^2}\ge\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}$$

$$\Leftrightarrow 4\sum_{cyc}(a+b)^2\ge\sum_{cyc}a^2+3\left(\sum_{cyc}a\right)^2+2\sqrt{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)$$

$$\overset{\text{via (1) and (4)}}{\Leftrightarrow} 4\sum_{cyc}x^2\ge s^2-8Rr-2r^2+3s^2+2\sqrt{3}s(s^2-8Rr-2r^2)$$

$$\Leftrightarrow 8(s^2-4Rr-r^2)\ge s^2-8Rr-2r^2+3s^2+2\sqrt{3}s(s^2-8Rr-2r^2)$$

$$\Leftrightarrow 2s^2-12Rr-3r^2\ge\sqrt{3}s(s^2-8Rr-2r^2)$$

$$\Leftrightarrow(2s^2-12Rr-3r^2)^2\overset{(*)}{\ge}3s^2(s^2-8Rr-2r^2)\text{ and }\because(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\ge}0\ \therefore\text{ in order to prove }(*),\text{ it suffices to prove :}$$

$$(2s^2-12Rr-3r^2)^2-3s^2(s^2-8Rr-2r^2)\ge(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(R-2r)s^2\ge r(14R^2-29Rr+2r^2)=r(R-2r)(14R-r)$$

$$\Leftrightarrow(R-2r)(s^2-14Rr+r^2)\ge 0\Leftrightarrow(R-2r)\left(s^2-16Rr+5r^2+2r(R-2r)\right)\ge 0$$

$$\to\text{true}\because R-2r\overset{\text{Euler}}{\ge}0\text{ and }s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0\Rightarrow(*)\text{ is true}$$

$$\therefore\sqrt{\sum_{cyc}(a+b)^2}\ge\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}\ \forall a,b,c>0,$$

$$''='' \text{ iff } a=b=c=3\ (\text{QED})$$

1410. If $x,y,z>0, x+y+z=xyz$ then:

$$\sqrt{x^2yz+yz}+\sqrt{y^2xz+xz}+\sqrt{z^2xy+xy}\le 2xyz$$

Proposed by Samed Ahmedov-Azerbaijan

Solution by proposer

$$\sqrt{x^2yz+yz}=\sqrt{x^2-x^2+x^2yz+yz}=\sqrt{x^2+x(xyz-x)+yz}=$$
$$=\sqrt{x^2+x(y+z)+yz}=\sqrt{(x+y)(x+z)}$$

İt can also be shown that $\sqrt{y^2xz+xz}=\sqrt{(x+y)(y+z)}$ and $\sqrt{z^2xy+xy}=\sqrt{(x+z)(y+z)}$ is true

$$\sqrt{(x+y)(x+z)}\overset{AM\ge GM}{\le}\frac{x+y+x+z}{2}=\frac{2x+y+z}{2}\quad(1)$$
$$\Rightarrow\sqrt{(x+y)(y+z)}\le\frac{2y+x+z}{2}\quad(2)\overset{AM\ge GM}{\le}$$
$$\sqrt{(x+z)(y+z)}\le\frac{2z+x+y}{2}\quad(3)$$

(1)+ (2)+ (3) $\Rightarrow \sqrt{x^2yz+yz}+\sqrt{y^2xz+xz}+\sqrt{z^2xy+xy}\le\frac{4(x+y+z)}{2}=\frac{4xyz}{2}=2xyz$

1411. If $a,b,c>0$, then :

$$B+2+\sqrt{2}\ge\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\ge 2+\sqrt{2}+A,\text{ where :}$$

$$A=\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{(a^2+b^2)^2}+\left(1+\frac{1}{2\sqrt{2}}\right)\cdot\frac{(a-b)^2}{a^2+b^2}\text{ and }B=\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{(a-b)^2}{ab}$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}+\frac{b}{a}-2-A$$

$$=\frac{(a-b)^2}{ab}-\left(1+\frac{1}{2\sqrt{2}}-1\right)\cdot\frac{(a-b)^4}{(a^2+b^2)^2}-\left(2-\left(1-\frac{1}{2\sqrt{2}}\right)\right)\cdot\frac{(a-b)^2}{a^2+b^2}$$

$$=\frac{(a-b)^2}{ab}-\frac{2(a-b)^2}{a^2+b^2}-\frac{(a-b)^4}{(a^2+b^2)^2}+\left(1-\frac{1}{2\sqrt{2}}\right)\cdot\left(\frac{(a-b)^4}{(a^2+b^2)^2}+\frac{(a-b)^2}{a^2+b^2}\right)$$

$$=(a-b)^2\left(\frac{1}{ab}-\frac{2}{a^2+b^2}\right)-\frac{(a-b)^4}{(a^2+b^2)^2}+\frac{\left(1-\frac{1}{2\sqrt{2}}\right)(a-b)^2}{a^2+b^2}\cdot\left(\frac{(a-b)^2}{a^2+b^2}+1\right)$$

$$=(a-b)^2\cdot\frac{(a-b)^2}{ab(a^2+b^2)}-\frac{(a-b)^4}{(a^2+b^2)^2}+\frac{2\left(1-\frac{1}{2\sqrt{2}}\right)(a-b)^2}{a^2+b^2}\cdot\frac{a^2+b^2-ab}{a^2+b^2}$$

$$=\frac{(a-b)^4(a^2+b^2-ab)}{ab(a^2+b^2)^2}+\frac{\left(2-\frac{1}{\sqrt{2}}\right)(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}$$

$$=\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\frac{(a-b)^2}{ab}+2-\frac{1}{\sqrt{2}}\right)$$

$$\therefore\frac{a}{b}+\frac{b}{a}-2-A\overset{(*)}{\ge}\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\frac{a^2+b^2}{ab}-\frac{1}{\sqrt{2}}\right)$$

$$\text{Again, }\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}}=\frac{\left(\sqrt{2(a^2+b^2)}-(a+b)\right)\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}{\sqrt{a^2+b^2}\cdot\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}$$

$$=\frac{2(a^2+b^2)-(a+b)^2}{\sqrt{2}.(a^2+b^2)+\sqrt{a^2+b^2}.(a+b)}=\frac{(a-b)^2}{\sqrt{2}.(a^2+b^2)+\sqrt{a^2+b^2}.(a+b)}$$

$$\le\frac{(a-b)^2}{\sqrt{2}.(a^2+b^2)+\frac{(a+b)^2}{\sqrt{2}}}\therefore\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}}\overset{(**)}{\le}\frac{(a-b)^2}{\sqrt{2}.(a^2+b^2)+\frac{(a+b)^2}{\sqrt{2}}}\therefore(*),(**)\Rightarrow$$

$$\text{in order to prove : }\frac{a}{b}+\frac{b}{a}-2-A\ge\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}},\text{ it suffices to prove :}$$

$$\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\frac{a^2+b^2}{ab}-\frac{1}{\sqrt{2}}\right)\geq\frac{(a-b)^2}{\sqrt{2}.(a^2+b^2)+\frac{(a+b)^2}{\sqrt{2}}}$$

$$\Leftrightarrow\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\sqrt{2}.\frac{a^2+b^2}{ab}-1\right)\geq\frac{\sqrt{2}.(a-b)^2}{\sqrt{2}.(a^2+b^2)+\frac{(a+b)^2}{\sqrt{2}}}$$

$$\Leftrightarrow\boxed{\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\sqrt{2}.\frac{a^2+b^2}{ab}-1\right)\overset{(\bullet)}{\geq}\frac{2(a-b)^2}{2(a^2+b^2)+(a+b)^2}}\text{ and}$$

$\because (a-b)^2\geq 0$ and $\sqrt{2}>1$, in order to prove $(\bullet)$, it suffices to prove :

$$\frac{(a^2+b^2-ab)\left(\frac{a^2+b^2}{ab}-1\right)}{(a^2+b^2)^2}\geq\frac{2}{2(a^2+b^2)+(a+b)^2}$$

$$\Leftrightarrow\left(2(a^2+b^2)+(a+b)^2\right)(a^2+b^2-ab)^2\geq 2ab(a^2+b^2)^2$$

$$\Leftrightarrow 3t^6-6t^5+8t^4-10t^3+8t^2-6t+3\geq 0\ \left(t=\frac{a}{b}\right)$$

$$\Leftrightarrow\boxed{(3t^4+5t^2+3)(t-1)^2\geq 0}\rightarrow\text{true}\Rightarrow(\bullet)\text{ is true}\therefore\frac{a}{b}+\frac{b}{a}-2-A$$

$$\geq\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}}\Rightarrow\boxed{\boxed{\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2}+A}}, ''='' \text{ iff } a=b$$

$$\text{Now, }B+2+\sqrt{2}\geq\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\Leftrightarrow\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{(a-b)^2}{ab}+2+\sqrt{2}$$

$$\geq\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\Leftrightarrow\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{a}{b}+\frac{b}{a}+\left(\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}}\right)\geq\frac{a}{b}+\frac{b}{a}$$

$$\Leftrightarrow\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{\left(\sqrt{2(a^2+b^2)}-(a+b)\right)\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}{\sqrt{a^2+b^2}.\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}\geq 0$$

$$\Leftrightarrow\boxed{\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{(a-b)^2}{\sqrt{a^2+b^2}.\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}\geq 0}\rightarrow\text{true}\because a,b>0$$

$$\therefore\boxed{\boxed{B+2+\sqrt{2}\geq\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}}}, ''='' \text{ iff } a=b$$

$$\therefore B+2+\sqrt{2}\geq\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2}+A\ \forall\, a,b>0, ''='' \text{ iff } a=b\ \text{(QED)}$$

1412. If $a,b,c>0$ with $p\leq 3$, then :

$$\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right)\geq\sum_{cyc}\frac{1}{a+b}\geq\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right),$$

where : $A=1+\frac{1}{8pq}\sum_{cyc}c(a-b)^2,\ p=a+b+c,\ q=ab+bc+ca$

When does equality holds ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3)\text{ and we have : }A=1+\frac{1}{8pq}\cdot\sum_{cyc}c(a-b)^2$$

$$=1+\frac{1}{8(\sum_{cyc}a)(\sum_{cyc}ab)}\cdot\sum_{cyc}\left(c(a^2+b^2-2ab)\right)$$

$$=1+\frac{1}{8(\sum_{cyc}a)(\sum_{cyc}ab)}\cdot\left(\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)-6abc\right)$$

$$=1+\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-9abc}{8(\sum_{cyc}a)(\sum_{cyc}ab)}\overset{\text{via (1),(2) and (3)}}{=}1+\frac{s(4Rr+r^2)-9r^2s}{8s(4Rr+r^2)}$$

$$\Rightarrow A=\frac{9R}{2(4R+r)}\to(4)\text{ and }\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right)\overset{3\ge p}{\ge}\frac{9}{8A}\left(\frac{1}{p}+\frac{p}{q}\right)=\frac{9}{8A}\left(\frac{1}{\sum_{cyc}a}+\frac{\sum_{cyc}a}{\sum_{cyc}ab}\right)$$

$$\overset{\text{via (1),(3) and (4)}}{=}\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\left(\frac{1}{s}+\frac{s}{4Rr+r^2}\right)=\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\frac{s^2+4Rr+r^2}{rs(4R+r)}$$

$$=\frac{s^2+4Rr+r^2}{4Rrs}=\frac{\sum_{cyc}xy}{xyz}=\sum_{cyc}\frac{1}{x}=\sum_{cyc}\frac{1}{a+b}\Rightarrow\boxed{\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right)\ge\sum_{cyc}\frac{1}{a+b}}$$

$$\text{Again, }\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)=\frac{9}{8A}\left(\frac{4}{3}+\frac{p}{q}+\frac{q}{p}-2\right)\le\frac{9}{8A}\left(\frac{4}{3}+\frac{p}{q}+1-2\right)$$

$$\left(\because p\le3\le\frac{(\sum_{cyc}a)^2}{\sum_{cyc}ab}=\frac{p^2}{q}\Rightarrow\frac{q}{p}\le1\right)=\frac{9}{8A}\left(\frac{p}{q}+\frac{1}{3}\right)\overset{\frac{1}{3}\le\frac{1}{p}}{\le}\frac{9}{8A}\left(\frac{p}{q}+\frac{1}{p}\right)$$

$$=\frac{9}{8A}\left(\frac{1}{\sum_{cyc}a}+\frac{\sum_{cyc}a}{\sum_{cyc}ab}\right)\overset{\text{via (1),(3) and (4)}}{=}\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\left(\frac{1}{s}+\frac{s}{4Rr+r^2}\right)$$

$$=\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\frac{s^2+4Rr+r^2}{rs(4R+r)}=\frac{s^2+4Rr+r^2}{4Rrs}=\frac{\sum_{cyc}xy}{xyz}=\sum_{cyc}\frac{1}{x}=\sum_{cyc}\frac{1}{a+b}$$

$$\therefore \boxed{\sum_{cyc}\frac{1}{a+b}\ge\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)}$$

$$\therefore \frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right)\ge\sum_{cyc}\frac{1}{a+b}\ge\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)\ \forall\, a,b,c>0\ with$$

$p\le 3$ where $p=a+b+c, q=ab+bc+ca,''=''$ iff $a=b=c=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\sum_{cyc}\frac{1}{a+b}=\frac{p^2+q}{(a+b)(b+c)(c+a)}$ and

$A=\frac{9(a+b)(b+c)(c+a)}{8pq}$, then the problem becomes to prove that

$$pq\left(\frac{1}{p}+\frac{3}{q}\right)\ge p^2+q\ge pq\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right),$$

We have $pq\left(\frac{1}{p}+\frac{3}{q}\right)=q+3p\ge q+p^2$ and

$$pq\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)=p^2+q^2-\frac{2pq}{3}\overset{?}{\le}p^2+q$$

$$\Leftrightarrow 3+2p\ge 3q\ \Leftrightarrow\ (3-p)(1+p)+(p^2-3q)\ge 0,$$

which is true because $p\le 3$ and $p^2\ge 3q$. So the proof is complete.

Equality holds iff $a=b=c=1$.

1413. If $a,b,c\ge 0$ then :

$$\frac{a^3}{(a+1)(b+1)}+\frac{b^3}{(b+1)(c+1)}+\frac{c^3}{(c+1)(a+1)}\ge\frac{3(a+b+c)(a^2+b^2+c^2)}{(a+b+c+3)^2}$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c$ and $q := a^2+b^2+c^2$. We have $3q \ge p^2$.

By AM − GM inequality, we have

$$3(a^2b+b^2c+c^2a) \le a^2b+b^2c+c^2a+(a^3+ab^2)+(b^3+bc^2)+(c^3+ca^2) = pq.$$

By CBS inequality, we have

$$\frac{a^3}{(a+1)(b+1)}+\frac{b^3}{(b+1)(c+1)}+\frac{c^3}{(c+1)(a+1)} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{cyc} a(a+1)(b+1)}$$

$$= \frac{q^2}{\sum_{cyc} a^2b+\sum_{cyc} a^2+\sum_{cyc} ab+\sum_{cyc} a} \ge \frac{q^2}{\frac{pq}{3}+\frac{p^2+q}{2}+p} \overset{?}{\ge} \frac{3pq}{(p+3)^2}.$$

$$\Leftrightarrow 2q(p+3)^2 \ge p(2pq+3p^2+3q+6p) \Leftrightarrow 3pq+6q \ge p^3+2p^2$$
$$\Leftrightarrow (3q-p^2)(p+2) \ge 0,$$

which is true and the proof is complete. Equality holds iff $a=b=c$.

1414. If $a, b, c > 0$ and $a^3+b^3+c^3 = 3$, then prove that :

$$\frac{1}{a^4+b^5+c^6}+\frac{1}{b^4+c^5+a^6}+\frac{1}{c^4+a^5+b^6} \le 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Via weighted AM − HM inequality, $\frac{a.a^3+b^2.b^3+c^3.c^3}{a^3+b^3+c^3}$

$$\ge \frac{a^3+b^3+c^3}{\frac{a^3}{a}+\frac{b^3}{b^2}+\frac{c^3}{c^3}} \overset{\because a^3+b^3+c^3=3}{\Rightarrow} a^4+b^5+c^6 \ge \frac{9}{a^2+b+1}$$

$$\Rightarrow \frac{1}{a^4+b^5+c^6} \le \frac{a^2+b+1}{9} \text{ and analogs}$$

$$\therefore \frac{1}{a^4+b^5+c^6}+\frac{1}{b^4+c^5+a^6}+\frac{1}{c^4+a^5+b^6} \le \sum_{cyc} \frac{a^2+b+1}{9}$$

$$\le \frac{1}{9}\left(\sum_{cyc} a^2+\sum_{cyc} a+3\right) \overset{?}{\le} 1 \Leftrightarrow \sum_{cyc} a^2+\sum_{cyc} a \underset{(*)}{\overset{?}{\le}} 6$$

$$\text{Now, via Power Mean Inequality,} \left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}} \geq \left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}$$

$$\overset{\because a^3+b^3+c^3=3}{\Rightarrow} 1 \geq \sqrt{\frac{\sum_{cyc} a^2}{3}} \Rightarrow \sum_{cyc} a^2 \leq 3 \to (1)$$

$$\text{Again, via Holder, } a^3+b^3+c^3 \geq \frac{1}{9}\left(\sum_{cyc} a\right)^3 \overset{\because a^3+b^3+c^3=3}{\Rightarrow} 3 \geq \frac{1}{9}\left(\sum_{cyc} a\right)^3$$

$$\Rightarrow \sum_{cyc} a \leq 3 \to (2) \therefore (1)+(2) \Rightarrow \text{LHS of } (*) \leq 3+3 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^4+b^5+c^6} + \frac{1}{b^4+c^5+a^6} + \frac{1}{c^4+a^5+b^6} \leq 1 \ \forall a,b,c>0 \,\middle|\, a^3+b^3+c^3=3,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1415. If $a,b,c,d>0$, *then prove that*:

$$\frac{a}{2023b+2024c+2025d} + \frac{b}{2023c+2024d+2025a} + \frac{c}{2023d+2024a+2025b} + \frac{d}{2023a+2024b+2025c} \geq \frac{1}{1518}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{2023b+2024c+2025d} + \frac{b}{2023c+2024d+2025a} + \frac{c}{2023d+2024a+2025b} + \frac{d}{2023a+2024b+2025c}$$

$$= \frac{a^2}{2023ab+2024ac+2025ad} + \frac{b^2}{2023bc+2024bd+2025ab} + \frac{c^2}{2023cd+2024ca+2025bc} + \frac{d^2}{2023ad+2024bd+2025cd}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{(a+b+c+d)^2}{4048(ab+ac+ad+bc+bd+cd)} \overset{?}{\geq} \frac{1}{1518}$$

$$\Leftrightarrow \frac{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}{(ab+ac+ad+bc+bd+cd)} \overset{?}{\geq} \frac{8}{3}$$

$$\Leftrightarrow 3(a^2+b^2+c^2+d^2) \underset{(*)}{\overset{?}{\geq}} 2(ab+ac+ad+bc+bd+cd)$$

$$\text{Now, } a^2+b^2+c^2 \geq ab+ac+bc \to (1), a^2+b^2+d^2 \geq ab+ad+bd \to (2),$$

$$a^2+c^2+d^2 \geq ac+ad+cd \to (3), b^2+c^2+d^2 \geq bc+bd+cd \to (4)$$

$$(1)+(2)+(3)+(4)\Rightarrow 3(a^2+b^2+c^2+d^2)\ge 2(ab+ac+ad+bc+bd+cd)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{a}{2023b+2024c+2025d}+\frac{b}{2023c+2024d+2025a}$$

$$+\frac{c}{2023d+2024a+2025b}+\frac{d}{2023a+2024b+2025c}\ge\frac{1}{1518}$$

$$\forall a,b,c,d>0,''='' \text{ iff } a=b=c=d \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Moroccco

By AM − GM inequality, we have

$$\frac{a}{2023b+2024c+2025d}+\frac{a(2023b+2024c+2025d)}{1518^2(a+b+c+d)^2}\ge\frac{a}{759(a+b+c+d)},$$

then

$$\frac{a}{2023b+2024c+2025d}\ge\frac{a}{759(a+b+c+d)}-\frac{a(2023b+2024c+2025d)}{1518^2(a+b+c+d)^2}$$

(and analogs)

Therefore

$$\sum_{cyc}\frac{a}{2023b+2024c+2025d}\ge\frac{1}{759}-\frac{4048(ab+bc+cd+da+ac+bd)}{1518^2(a+b+c+d)^2}$$

$$\overset{Maclaurin}{\ge}\frac{1}{759}-\frac{4048}{1518^2}\cdot\frac{3}{8}=\frac{1}{1518},$$

as desired. Equality holds iff $a=b=c=d$.

1416. **If $a,b,c,d>0$ and $n\in\mathbb{N}$, then prove that :**

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}$$

$$+\frac{c}{nd+(n+1)a+(n+2)b}+\frac{d}{na+(n+1)b+(n+2)c}\ge\frac{4}{3(n+1)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}$$
$$+\frac{c}{nd+(n+1)a+(n+2)b}+\frac{d}{na+(n+1)b+(n+2)c}$$
$$=\frac{a^2}{nab+(n+1)ac+(n+2)ad}+\frac{b^2}{nbc+(n+1)bd+(n+2)ab}$$
$$+\frac{c^2}{ncd+(n+1)ca+(n+2)bc}+\frac{d^2}{nad+(n+1)bd+(n+2)cd}\overset{Bergstrom}{\ge}$$

$$\frac{(a+b+c+d)^2}{2(n+1)(ab+ac+ad+bc+bd+cd)} \overset{?}{\geq} \frac{4}{3(n+1)}$$

$$\Leftrightarrow \frac{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}{ab+ac+ad+bc+bd+cd} \overset{?}{\geq} \frac{8}{3}$$

$$\Leftrightarrow 3(a^2+b^2+c^2+d^2) \underbrace{\overset{?}{\geq}}_{(*)} 2(ab+ac+ad+bc+bd+cd)$$

Now, $a^2+b^2+c^2 \geq ab+ac+bc \to (1), a^2+b^2+d^2 \geq ab+ad+bd \to (2),$
$a^2+c^2+d^2 \geq ac+ad+cd \to (3), b^2+c^2+d^2 \geq bc+bd+cd \to (4)$

$$(1)+(2)+(3)+(4) \Rightarrow 3(a^2+b^2+c^2+d^2) \geq 2(ab+ac+ad+bc+bd+cd)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{a}{nb+(n+1)c+(n+2)d} + \frac{b}{nc+(n+1)d+(n+2)a}$$

$$+\frac{c}{nd+(n+1)a+(n+2)b} + \frac{d}{na+(n+1)b+(n+2)c} \geq \frac{4}{3(n+1)}$$

$\forall a,b,c,d > 0, '' ='' \text{ iff } a=b=c=d$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{a}{nb+(n+1)c+(n+2)d} + \frac{16a[nb+(n+1)c+(n+2)d]}{9(n+1)^2(a+b+c+d)^2} \geq \frac{8a}{3(n+1)(a+b+c+d)},$$

then

$$\frac{a}{nb+(n+1)c+(n+2)d} \geq \frac{8a}{3(n+1)(a+b+c+d)} - \frac{16[nab+(n+1)ac+(n+2)ad]}{9(n+1)^2(a+b+c+d)^2}$$

Adding this inequality with the similar ones, we obtain

$$\sum_{cyc} \frac{a}{nb+(n+1)c+(n+2)d} \geq \frac{8}{3(n+1)} - \frac{32}{9(n+1)} \cdot \frac{ab+bc+cd+da+ac+bd}{(a+b+c+d)^2}$$

$$\underbrace{\geq}_{Maclaurin} \frac{8}{3(n+1)} - \frac{32}{9(n+1)} \cdot \frac{3}{8} = \frac{4}{3(n+1)},$$

as desired. Equality holds iff $a=b=c=d$.

1417. If $a,b,c > 0$ and $a^n+b^n+c^n = 3$, then prove that :

$$\frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}} + \frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}} + \frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}} \leq 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall x,y,z > 0, (x^2b+a^2y^2+z^2a^2b)(a^2+b+1) \overset{?}{\geq} a^2b(x+y+z)^2$$

$$\Leftrightarrow (a^4bz^2 - 2a^2bxz + bx^2) + (a^2b^2z^2 - 2a^2byz + a^2y^2)$$

$$+(a^4y^2 - 2a^2bxy + b^2x^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow b\left(a^2z-x\right)^2+a^2(bz-y)^2+\left(a^2y-bx\right)^2\overset{?}{\geq}0\rightarrow \text{true}$$

$$\therefore \boxed{\left(x^2b+a^2y^2+z^2a^2b\right)\left(a^2+b+1\right)\geq a^2b(x+y+z)^2}\rightarrow(1)$$

$$\text{Also, }\left(x^2b^2c+y^2c+z^2b^2\right)\left(b^2+c+1\right)\overset{?}{\geq}b^2c(x+y+z)^2$$

$$\Leftrightarrow \left(b^4cx^2-2b^2cxy+cy^2\right)+\left(b^2c^2x^2-2b^2cxz+b^2z^2\right)$$

$$+\left(b^4z^2-2b^2cyz+c^2y^2\right)\overset{?}{\geq}0$$

$$\Leftrightarrow c\left(b^2x-y\right)^2+b^2(cx-z)^2+\left(b^2z-cy\right)^2\overset{?}{\geq}0\rightarrow \text{true}$$

$$\therefore \boxed{\left(x^2b^2c+y^2c+z^2b^2\right)\left(b^2+c+1\right)\geq b^2c(x+y+z)^2}\rightarrow(2)$$

$$\text{Again, }\left(y^2c^2a+z^2a+x^2c^2\right)\left(c^2+a+1\right)\overset{?}{\geq}c^2a(x+y+z)^2$$

$$\Leftrightarrow \left(c^4ay^2-2c^2ayz+az^2\right)+\left(c^2a^2y^2-2c^2axy+c^2x^2\right)$$

$$+\left(c^4x^2-2c^2axz+a^2z^2\right)\overset{?}{\geq}0$$

$$\Leftrightarrow a\left(c^2y-z\right)^2+c^2(ay-x)^2+\left(c^2x-az\right)^2\overset{?}{\geq}0\rightarrow \text{true}$$

$$\therefore \boxed{\left(y^2c^2a+z^2a+x^2c^2\right)\left(c^2+a+1\right)\geq c^2a(x+y+z)^2}\rightarrow(3)$$

$$\text{Putting } x=a^n, y=b^n, z=c^n \text{ in } (1), \text{ we arrive at: } \frac{a^{2n}.b+a^2.b^{2n}+c^{2n}a^2b}{a^2b}$$

$$\geq\frac{9}{a^2+b+1}\ (\because x+y+z=3)\Rightarrow\frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}}\leq\frac{a^2+b+1}{9}\rightarrow(\text{i})$$

$$\text{Putting } x=a^n, y=b^n, z=c^n \text{ in } (2), \text{ we arrive at: } \frac{a^{2n}.b^2c+c.b^{2n}+c^{2n}.b^2}{b^2c}$$

$$\geq\frac{9}{b^2+c+1}\ (\because x+y+z=3)\Rightarrow\frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}}\leq\frac{b^2+c+1}{9}\rightarrow(\text{ii})$$

$$\text{Putting } x=a^n, y=b^n, z=c^n \text{ in } (3), \text{ we arrive at: } \frac{a^{2n}.c^2+b^{2n}.c^2a+c^{2n}.a}{c^2a}$$

$$\geq\frac{9}{c^2+a+1}\ (\because x+y+z=3)\Rightarrow\frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}}\leq\frac{c^2+a+1}{9}\rightarrow(\text{iii})$$

$$\therefore (\text{i})+(\text{ii})+(\text{iii})\Rightarrow$$

$$\frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}}+\frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}}+\frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}}$$

$$\leq\sum_{cyc}\frac{a^2+b+1}{9}=\frac{1}{9}\left(\sum_{cyc}a^2+\sum_{cyc}a+3\right)\overset{?}{\leq}1\Leftrightarrow\sum_{cyc}a^2+\sum_{cyc}a\underset{(*)}{\overset{?}{\leq}}6$$

$$\text{Now, via Power Mean Inequality, }\left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}}\geq\left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}$$

$$\overset{\because a^3+b^3+c^3=3}{\Rightarrow}1\geq\sqrt{\frac{\sum_{cyc}a^2}{3}}\Rightarrow\sum_{cyc}a^2\leq3\rightarrow(\bullet)$$

$$\text{Again, via Holder, } a^3+b^3+c^3\geq\frac{1}{9}\left(\sum_{cyc}a\right)^3\overset{\because a^3+b^3+c^3=3}{\Rightarrow}3\geq\frac{1}{9}\left(\sum_{cyc}a\right)^3$$

$$\Rightarrow \sum_{cyc} a \le 3 \to (\bullet\bullet) \therefore (\bullet) + (\bullet\bullet) \Rightarrow \text{LHS of } (*) \le 3 + 3 = 6 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^{2n-2} + b^{2n-1} + c^{2n}} + \frac{1}{b^{2n-2} + c^{2n-1} + a^{2n}} + \frac{1}{c^{2n-2} + a^{2n-1} + b^{2n}} \le 1$$

$$\forall a, b, c > 0 \mid a^n + b^n + c^n = 3, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1418. If $x, y, z > 0, x + y + z = 1$ then:

$$\sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \le 2$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$LHS = \sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \overset{X+Y+Z=1}{\cong}$$

$$\sqrt{x(x + y + z) + yz} + \sqrt{y(x + y + z) + xz} + \sqrt{z(x + y + z) + xy}$$

$$= \sqrt{x^2 + xy + xz + yz} + \sqrt{y^2 + xy + xz + yz} + \sqrt{z^2 + xz + xy + yz}$$

$$= \sqrt{x(x + z) + y(x + z)} + \sqrt{x(y + z) + y(y + z)} + \sqrt{x(y + z) + z(y + z)}$$

$$= \sqrt{(x + z)(x + y)} + \sqrt{(y + z)(x + y)} + \sqrt{(y + z)(x + z)}$$

$$\overset{AM-GM}{\cong} \frac{2x + y + z}{2} + \frac{2y + x + z}{2} + \frac{2z + x + y}{2} = 2(x + y + z) = 2.1 = 2 = RHS$$

$$Hence: \sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \le 2, = iff: x = y = z = \frac{1}{3}$$

Solution 2 by Sakirin Ly-Cambodgia

$$Method\ 1: using\ AM - GM$$

$$we - get: \sqrt{\frac{4}{9}}S = \sqrt{\frac{4}{9}(x + yz)} + \sqrt{\frac{4}{9}(y + zx)} + \sqrt{\frac{4}{9}(z + xy)}$$

$$\frac{2}{3}S \le \frac{\frac{4}{9} + (x + yz)}{2} + \frac{\frac{4}{9} + (y + zx)}{2} + \frac{\frac{4}{9} + (z + xy)}{2}$$

$$\frac{2}{3}S \le \frac{\frac{4}{3} + (x + y + z) + (xy + yz + zx)}{2} \le \frac{\frac{4}{3} + 1 + \frac{(x + y + z)^2}{3}}{2} = \frac{\frac{5}{3} + 1}{2} = \frac{4}{3}$$

$$\to S \le \frac{4}{3}.\frac{3}{2} = 2 \to S \le 2 \ \left(Equalities - when: x = y = z = \frac{1}{3}\right)$$

$$Method2: using\ C - B - S$$

$$S^2 = (\sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy})^2 \le (1 + 1 + 1)(x + y + z + xy + yz + zx)$$

$$\le 3\left(1 + \frac{(x + y + z)^2}{3}\right) \to S^2 \le 3.\frac{4}{3} = 4 => S \le 2$$

$$(Equalities - when: x = y = z = \frac{1}{3})$$

Solution 3 by Samed Ahmedov-Azerbaijan

$$x = 1 - (y + z) => \sqrt{x + yz} = \sqrt{1 - (y + z) + yz}$$
$$= \sqrt{(1 - y)(1 - z)} \overset{AM-GM}{\leq} \frac{2 - (y + z)}{2}$$
$$also \sqrt{y + xz} = \sqrt{(1 - x)(1 - z)} \leq \frac{2 - (x + z)}{2}$$
$$\sqrt{z + xy} = \sqrt{(1 - x)(1 - y)} \leq \frac{2 - (x + y)}{2}$$
$$\sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \leq \frac{6 - 2(x + y + z)}{2} = \frac{4}{2} = 2$$
$$Equalities - when: x = y = z = \frac{1}{3}$$

1419.

If $\{x, y, z\}$ be non – negative real nmbers such that: $x + y + z = 3$,

then prove that: $\frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ Exactly 1 variable equals to zero and WLOG we may assume

$$x = 0\ (y + z = 3 \text{ with } y, z > 0) \text{ and then}: \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$$
$$\Leftrightarrow \frac{6}{1 + yz} + 9 \leq 4((y + z)^2 - 2yz) \overset{y+z=3}{\Leftrightarrow} \frac{6}{1 + t} + 9 \leq 4(9 - 2t)\ (t = yz)$$
$$\Leftrightarrow 8t^2 - 19t - 21 \geq 0 \Leftrightarrow t \geq \frac{19 - \sqrt{1033}}{16} \approx -0.82127 \text{ and } t \leq \frac{19 - \sqrt{1033}}{16}$$
$$\approx 3.19627 \to \text{true} \because t \in \left(0, \frac{9}{4}\right] \left(\because 3 = y + z \overset{A-G}{\geq} 2.\sqrt{yz} \Rightarrow \sqrt{t} \leq \frac{3}{2} \Rightarrow t \leq \frac{9}{4}\right)$$
$$\therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} < 6$$

$\boxed{Case\ 2}$ Exactly 2 variables equal to zero and WLOG we may assume $y = z = 0$

$$(x = 3) \text{ and then}: \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6 \Leftrightarrow \frac{12 - 8.9}{1} + 12 + 12$$
$$= -36 < 6 \therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} < 6$$

Case 3 $x, y, z > 0$ and then : $\frac{12-8x^2}{1+yz}+\frac{12-8y^2}{1+xz}+\frac{12-8z^2}{1+xy}\le 6$

$$\Leftrightarrow 6\sum_{cyc}\frac{1}{1+yz}\le 4\sum_{cyc}\frac{x^2}{1+yz}+3\to(\text{i})$$

$$\sum_{cyc}\frac{1}{1+yz}=\frac{\sum_{cyc}\left((1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)}=\frac{xyz\sum_{cyc}x+2\sum_{cyc}xy+3}{x^2y^2z^2+xyz\sum_{cyc}x+\sum_{cyc}xy+1}$$

$$\overset{x+y+z=3}{=}\frac{3xyz+2\sum_{cyc}xy+3}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}\Rightarrow 6\sum_{cyc}\frac{1}{1+yz}-3$$

$$=\frac{18xyz+12\sum_{cyc}xy+18}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}-3$$

$$\Rightarrow 6\sum_{cyc}\frac{1}{1+yz}-3\overset{(*)}{=}\frac{9xyz+9\sum_{cyc}xy+15-3x^2y^2z^2}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}\ \textit{and}$$

$$4\sum_{cyc}\frac{x^2}{1+yz}=\frac{4\sum_{cyc}\left(x^2(1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)}$$

$$=\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-4xyz\sum_{cyc}x+4\sum_{cyc}x^2}{x^2y^2z^2+xyz\sum_{cyc}x+\sum_{cyc}xy+1}$$

$$\overset{x+y+z=3}{=}\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}$$

$$\therefore 4\sum_{cyc}\frac{x^2}{1+yz}\overset{(**)}{=}\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}$$

$$\therefore (*),(**)\Rightarrow(\text{i})\Leftrightarrow 4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2$$

$$\ge 9xyz+9\sum_{cyc}xy+15-3x^2y^2z^2\overset{x+y+z=3}{\Leftrightarrow}$$

$$4xyz\sum_{cyc}x^3+\frac{4}{9}\cdot\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-\frac{12xyz}{27}\cdot\left(\sum_{cyc}x\right)^3$$

$$+\frac{4}{81}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)^4\ge\frac{9xyz}{27}\cdot\left(\sum_{cyc}x\right)^3$$

$$+\frac{9}{81}\cdot\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^4+\frac{15}{729}\cdot\left(\sum_{cyc}x\right)^6-3x^2y^2z^2$$

$$\Leftrightarrow \boxed{\begin{array}{l} 972xyz\sum_{cyc}x^3+108\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-189xyz\left(\sum_{cyc}x\right)^3+ \\ 12\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)^4\overset{(\bullet)}{\geq}27\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^4+5\left(\sum_{cyc}x\right)^6-729x^2y^2z^2\end{array}}$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(1)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$$\Rightarrow xyz\overset{(2)}{=}r^2s$$

Via such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2$

$$\Rightarrow\sum_{cyc}xy\overset{(3)}{=}4Rr+r^2\Rightarrow\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}x^2\overset{(4)}{=}s^2-8Rr-2r^2 \text{ and}$$

$$\sum_{cyc}x^3=\left(\sum_{cyc}x\right)^3-3(x+y)(y+z)(z+x)\overset{\text{via (1)}}{=}s^3-3.4Rrs$$

$$\Rightarrow\sum_{cyc}x^3\overset{(5)}{=}s(s^2-12Rr)\therefore \text{via (1),(2),(3),(4) and (5)}, (\bullet)\Leftrightarrow$$

$$972r^2s^2(s^2-12Rr)+108(s^2-8Rr-2r^2)(4Rr+r^2)s^2-189r^2s^4$$
$$+12(s^2-8Rr-2r^2)s^4\geq27(4Rr+r^2)s^4+5s^6-729r^4s^2$$

$$\Leftrightarrow\boxed{7s^4+rs^2(228R+840r)\overset{(\bullet\bullet)}{\geq}r(3456R^2+13392Rr-513r^2)}$$

Now, LHS of $(\bullet\bullet)\overset{\text{Gerretsen}}{\geq}\left(7(16Rr-5r^2)+r(228R+840r)\right)s^2\overset{\text{Gerretsen}}{\geq}$

$$(340Rr+805r^2)(16Rr-5r^2)\overset{?}{\geq}r(3456R^2+13392Rr-513r^2)$$

$$\Leftrightarrow 496R^2-553Rr-878r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(496R+439r)\overset{?}{\geq}0\rightarrow \text{true}$$

$\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$ is true $\therefore \frac{12-8x^2}{1+yz}+\frac{12-8y^2}{1+xz}+\frac{12-8z^2}{1+xy}\leq 6$ and combining all cases, $\frac{12-8x^2}{1+yz}+\frac{12-8y^2}{1+xz}+\frac{12-8z^2}{1+xy}\leq 6$

$\forall$ non $-$ negative $\{x,y,z\}$ such that : $x+y+z=3$, " = " iff $x=y=z=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since: $\frac{12-8x^2}{1+yz}=12-\frac{4(2x^2+3yz)}{1+yz}$ (and analogs),

then the desired inequality is equivalent to

$$\frac{2x^2+3yz}{1+yz}+\frac{2y^2+3zx}{1+zx}+\frac{2z^2+3xy}{1+xy}\geq\frac{15}{2}.$$

Let $p:=x+y+z=3,\ \ q:=xy+yz+zx,\ \ r:=xyz$. By AM – GM inequality, we have

$$\frac{2x^2+3yz}{1+yz}+\frac{(2x^2+3yz)(1+yz)}{4}\geq 2x^2+3yz\Rightarrow\frac{2x^2+3yz}{1+yz}\geq\frac{6x^2+9yz-2x^2yz-3y^2z^2}{4}.$$

$$\Rightarrow\sum_{cyc}\frac{2x^2+3yz}{1+yz}\geq\sum_{cyc}\frac{6x^2+9yz-2x^2yz-3y^2z^2}{4}=\frac{6(p^2-2q)+9q-2pr-3(q^2-2pr)}{4}$$

$$\overset{p=3}{=}\frac{54+12r-3q-3q^2}{4}\overset{?}{\geq}\frac{15}{2}\Leftrightarrow 8+4r\geq q+q^2.$$

If $q\leq 2$, we have $q+q^2\leq 2+2^2<8\leq 8+4r$.

If $2\leq q$. Since $3q\leq p^2=9$, then $q\leq 3$, and by Schur's inequality, we have

$$8+4r\geq 8+4.\frac{p(4q-p^2)}{9}=8+\frac{4(4q-9)}{3}=q+q^2+\frac{(3-q)(3q-4)}{3}\geq q+q^2.$$

So the proof is complete. Equality holds iff $x=y=z=1$.

1420.

Let $\{x,y,z\}$ be positive real nmbers such that : $xyz=1$. Prove that :

$$\frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}\geq 6$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}=\sum_{cyc}\frac{x+1+6}{(x+1)^2}$$

$$=\sum_{cyc}\frac{1}{x+1}+6\left(\left(\sum_{cyc}\frac{1}{x+1}\right)^2-2\sum_{cyc}\frac{1}{(y+1)(z+1)}\right)$$

$$=\frac{3+2m+n}{2+m+n}+6\left(\left(\frac{3+2m+n}{2+m+n}\right)^2-\frac{2(m+3)}{2+m+n}\right)$$

$$\left(m=\sum_{cyc}x,n=\sum_{cyc}xy\text{ and }\because xyz=1\right)\overset{?}{\geq}6$$

$$\Leftrightarrow 8m^2+3mn+n^2-5m-19n-36\underset{(*)}{\overset{?}{\geq}}0$$

Case 1 $m\geq n$ and then : LHS of $(*)=8m^2-5m-57+3mn+n^2-19n+21$

$$\overset{m\geq n}{\geq}(8m+19)(m-3)+4n^2-19n+21$$

$$=(8m+19)(m-3)+(4n-7)(n-3)\geq 0$$

$$\because m=\sum_{cyc}x\overset{A-G}{\geq}3.\sqrt[3]{xyz}\overset{xyz=1}{=}3\text{ and }n=\sum_{cyc}xy\overset{A-G}{\geq}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{=}3$$

$$\Rightarrow m,n\geq 3\Rightarrow(*)\text{ is true}$$

Case 2 $n\geq m$ and then : LHS of $(*)=8m^2-5m-36+3mn+n^2-19n\overset{n\geq m}{\geq}$

$$8m^2-5m-36+3m^2+m^2-19n\overset{m^2\geq 3n}{\geq}8m^2-5m-36+12n-19n$$

$$\overset{m^2\geq 3n}{\geq}8m^2-5m-36-\frac{7m^2}{3}=\frac{17m^2-15m-108}{3}=\frac{(17m+36)(m-3)}{3}\geq 0$$

$$\because m=\sum_{cyc}x\overset{A-G}{\geq}3.\sqrt[3]{xyz}\overset{xyz=1}{=}3\Rightarrow m\geq 3\Rightarrow(*)\text{ is true}\therefore\text{ combining both cases,}$$

$(*)$ is true $\forall\, x,y,z>0$ such that : $xyz=1$

$$\therefore\frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}\geq 6\ \forall\, x,y,z>0$$

such that : $xyz=1$, " = " iff $x=y=z=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $xyz=1$, we may put $x=\frac{bc}{a^2}$, $y=\frac{ca}{b^2}$, $z=\frac{ab}{c^2}$, where a,b,c are some positive real numbers. Then, the desired inequality becomes

$$\frac{7a^4+a^2bc}{(a^2+bc)^2}+\frac{7b^4+ab^2c}{(b^2+ca)^2}+\frac{7c^4+abc^2}{(c^2+ab)^2}\geq 6.$$

By the Cauchy Schwarz Inequality, we have

$$\sum_{cyc}\frac{7a^4+a^2bc}{(a^2+bc)^2}\geq\sum_{cyc}\frac{7a^4+a^2bc}{(a^2+b^2)(a^2+c^2)}=\frac{\sum_{cyc}(7a^4+a^2bc)(b^2+c^2)}{(a^2+b^2)(b^2+c^2)(a^2+c^2)}\overset{?}{\geq}6$$

$$\Leftrightarrow\sum_{cyc}(7a^4+a^2bc)(b^2+c^2)\geq 6(a^2+b^2)(b^2+c^2)(a^2+c^2)$$

$$\Leftrightarrow \sum_{cyc} a^4(b^2+c^2)+abc\sum_{cyc} a(b^2+c^2)\ge 12a^2b^2c^2 \qquad (1)$$

which is true by $AM-GM$ inequality:

$$LHS_{(1)}\ge \sum_{cyc} a^4.2bc+abc\sum_{cyc} a.2bc\ge 2abc.3abc+6a^2b^2c^2=12a^2b^2c^2.$$

Equality holds iff $x=y=z=1$.

1421.

If $\{x,y,z\}$ be non – negative real numbers such that : $x+y+z=3$,

then prove that : $\dfrac{12-8x^2}{1+yz}+\dfrac{12-8y^2}{1+xz}+\dfrac{12-8z^2}{1+xy}\le 6$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ Exactly 1 variable equals to zero and WLOG we may assume

$x=0$ ($y+z=3$ with $y,z>0$) and then : $\dfrac{12-8x^2}{1+yz}+\dfrac{12-8y^2}{1+xz}+\dfrac{12-8z^2}{1+xy}\le 6$

$$\Leftrightarrow \frac{6}{1+yz}+9\le 4\left((y+z)^2-2yz\right)\overset{y+z=3}{\Leftrightarrow}\frac{6}{1+t}+9\le 4(9-2t)\ (t=yz)$$

$$\Leftrightarrow 8t^2-19t-21\ge 0\Leftrightarrow t\ge\frac{19-\sqrt{1033}}{16}\approx -0.82127 \text{ and } t\le\frac{19-\sqrt{1033}}{16}$$

$$\approx 3.19627\to \text{true} \because t\in\left(0,\frac{9}{4}\right]\ \left(\because 3=y+z\overset{A-G}{\ge}2.\sqrt{yz}\Rightarrow\sqrt{t}\le\frac{3}{2}\Rightarrow t\le\frac{9}{4}\right)$$

$$\therefore \frac{12-8x^2}{1+yz}+\frac{12-8y^2}{1+xz}+\frac{12-8z^2}{1+xy}<6$$

$\boxed{Case\ 2}$ Exactly 2 variables equal to zero and WLOG we may assume $y=z=0$

$(x=3)$ and then : $\dfrac{12-8x^2}{1+yz}+\dfrac{12-8y^2}{1+xz}+\dfrac{12-8z^2}{1+xy}\le 6\Leftrightarrow\dfrac{12-8.9}{1}+12+12$

$$=-36<6\ \therefore \frac{12-8x^2}{1+yz}+\frac{12-8y^2}{1+xz}+\frac{12-8z^2}{1+xy}<6$$

$\boxed{Case\ 3}$ $x,y,z>0$ and then : $\dfrac{12-8x^2}{1+yz}+\dfrac{12-8y^2}{1+xz}+\dfrac{12-8z^2}{1+xy}\le 6$

$$\Leftrightarrow 6\sum_{cyc}\frac{1}{1+yz}\le 4\sum_{cyc}\frac{x^2}{1+yz}+3\to \text{(i)}$$

$$\sum_{cyc}\frac{1}{1+yz}=\frac{\sum_{cyc}\left((1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)}=\frac{xyz\sum_{cyc}x+2\sum_{cyc}xy+3}{x^2y^2z^2+xyz\sum_{cyc}x+\sum_{cyc}xy+1}$$

$$\overset{x+y+z=3}{=}\frac{3xyz+2\sum_{cyc}xy+3}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}\Rightarrow 6\sum_{cyc}\frac{1}{1+yz}-3$$

$$= \frac{18xyz + 12\sum_{cyc} xy + 18}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1} - 3$$

$$\Rightarrow 6\sum_{cyc} \frac{1}{1+yz} - 3 \overset{(*)}{=} \frac{9xyz + 9\sum_{cyc} xy + 15 - 3x^2y^2z^2}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1} \text{ and}$$

$$4\sum_{cyc} \frac{x^2}{1+yz} = \frac{4\sum_{cyc}\left(x^2(1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)}$$

$$= \frac{4xyz\sum_{cyc} x^3 + 4(\sum_{cyc} x^2)(\sum_{cyc} xy) - 4xyz\sum_{cyc} x + 4\sum_{cyc} x^2}{x^2y^2z^2 + xyz\sum_{cyc} x + \sum_{cyc} xy + 1}$$

$$\overset{x+y+z=3}{=} \frac{4xyz\sum_{cyc} x^3 + 4(\sum_{cyc} x^2)(\sum_{cyc} xy) - 12xyz + 4\sum_{cyc} x^2}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1}$$

$$\therefore 4\sum_{cyc} \frac{x^2}{1+yz} \overset{(**)}{=} \frac{4xyz\sum_{cyc} x^3 + 4(\sum_{cyc} x^2)(\sum_{cyc} xy) - 12xyz + 4\sum_{cyc} x^2}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1}$$

$$\therefore (*), (**) \Rightarrow (\mathrm{i}) \Leftrightarrow 4xyz\sum_{cyc} x^3 + 4\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) - 12xyz + 4\sum_{cyc} x^2$$

$$\ge 9xyz + 9\sum_{cyc} xy + 15 - 3x^2y^2z^2 \overset{x+y+z=3}{\Leftrightarrow}$$

$$4xyz\sum_{cyc} x^3 + \frac{4}{9}\cdot\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - \frac{12xyz}{27}\cdot\left(\sum_{cyc} x\right)^3$$

$$+\frac{4}{81}\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)^4 \ge \frac{9xyz}{27}\cdot\left(\sum_{cyc} x\right)^3$$

$$+\frac{9}{81}\cdot\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^4 + \frac{15}{729}\cdot\left(\sum_{cyc} x\right)^6 - 3x^2y^2z^2$$

$$\Leftrightarrow \boxed{\begin{array}{l} 972xyz\sum_{cyc} x^3 + 108\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 189xyz\left(\sum_{cyc} x\right)^3 + \\ 12\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)^4 \overset{(\bullet)}{\ge} 27\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^4 + 5\left(\sum_{cyc} x\right)^6 - 729x^2y^2z^2 \end{array}}$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(1)}{=} s \Rightarrow x = s-a, y = s-b, z = s-c$

$$\Rightarrow xyz \overset{(2)}{=} r^2 s$$

$$\text{Via such substitutions}, \sum_{cyc} xy = \sum_{cyc} (s-a)(s-b) = 4Rr + r^2$$

$$\Rightarrow \sum_{cyc} xy \overset{(3)}{=} 4Rr + r^2 \Rightarrow \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} x^2 \overset{(4)}{=} s^2 - 8Rr - 2r^2 \text{ and}$$

$$\sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x+y)(y+z)(z+x) \overset{\text{via (1)}}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} x^3 \overset{(5)}{=} s(s^2 - 12Rr) \therefore \text{via (1), (2), (3), (4) and (5)}, (\bullet) \Leftrightarrow$$

$$972r^2s^2(s^2 - 12Rr) + 108(s^2 - 8Rr - 2r^2)(4Rr + r^2)s^2 - 189r^2s^4$$
$$+12(s^2 - 8Rr - 2r^2)s^4 \geq 27(4Rr + r^2)s^4 + 5s^6 - 729r^4s^2$$

$$\Leftrightarrow \boxed{7s^4 + rs^2(228R + 840r) \overset{(\bullet\bullet)}{\geq} r(3456R^2 + 13392Rr - 513r^2)}$$

$$\text{Now, LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} \left(7(16Rr - 5r^2) + r(228R + 840r)\right)s^2 \overset{\text{Gerretsen}}{\geq}$$

$$(340Rr + 805r^2)(16Rr - 5r^2) \overset{?}{\geq} r(3456R^2 + 13392Rr - 513r^2)$$

$$\Leftrightarrow 496R^2 - 553Rr - 878r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(496R + 439r) \overset{?}{\geq} 0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6 \text{ and}$$

$$\text{combining all cases}, \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$$

$$\forall \text{ non-negative } \{x, y, z\} \text{ such that} : x + y + z = 3, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

1422. If $\{x, y, z\} \in \mathbb{R}^+$ such that : $xyz = 1$, then prove that :

$$\frac{x}{x^2 + 2} + \frac{y}{y^2 + 2} + \frac{z}{z^2 + 2} \leq 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{x^2 + 2} + \frac{y}{y^2 + 2} + \frac{z}{z^2 + 2} = \frac{\sum_{cyc}\left(x(y^2 + 2)(z^2 + 2)\right)}{(x^2 + 2)(y^2 + 2)(z^2 + 2)}$$

$$= \frac{\sum_{cyc}\left(x(2y^2 + 2z^2 + y^2z^2 + 4)\right)}{x^2y^2z^2 + 8 + 2\sum_{cyc} x^2y^2 + 4\sum_{cyc} x^2}$$

$$= \frac{2(\sum_{cyc} x)(\sum_{cyc} xy) - 6xyz + xyz\sum_{cyc} xy + 4\sum_{cyc} x}{x^2y^2z^2 + 8 + 2\sum_{cyc} x^2y^2 + 4\sum_{cyc} x^2} \leq 1$$

$$\overset{xyz=1}{\Leftrightarrow} \boxed{15+2\sum_{cyc}x^2y^2+4\sum_{cyc}x^2 \overset{(*)}{\geq} 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+\sum_{cyc}xy+4\sum_{cyc}x}$$

Now, $x^2+x^2y^2 \overset{A-G}{\geq} 2x^2y, y^2+y^2z^2 \overset{A-G}{\geq} 2y^2z, z^2+z^2x^2 \overset{A-G}{\geq} 2z^2x,$
$y^2+x^2y^2 \overset{A-G}{\geq} 2xy^2, z^2+y^2z^2 \overset{A-G}{\geq} 2yz^2, x^2+z^2x^2 \overset{A-G}{\geq} 2zx^2$
and summing up, we arrive at :

$$2\sum_{cyc}x^2+2\sum_{cyc}x^2y^2 \geq 2\sum_{cyc}x^2y+2\sum_{cyc}xy^2 = 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6xyz$$

$$\therefore 2\sum_{cyc}x^2+2\sum_{cyc}x^2y^2+6 \geq 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \to \text{(i)} \therefore \text{(i)} \Rightarrow \text{in order}$$

to prove $(*)$, it suffices to prove : $9+2\sum_{cyc}x^2 \geq \sum_{cyc}xy+4\sum_{cyc}x$

$$\overset{xyz=1}{\Leftrightarrow} \boxed{2\sum_{cyc}x^2-\sum_{cyc}xy+9.\sqrt[3]{x^2y^2z^2} \overset{(**)}{\geq} 4\left(\sum_{cyc}x\right).\sqrt[3]{xyz}}$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s \Rightarrow \sum_{cyc}x \overset{(1)}{=} s \Rightarrow x=s-a, y=s-b, z=s-c$

$\Rightarrow xyz \overset{(2)}{=} r^2s$ and via such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)$

$$=4Rr+r^2 \Rightarrow \sum_{cyc}xy \overset{(3)}{=} 4Rr+r^2 \Rightarrow \sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy$$

$$\overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc}x^2 \overset{(4)}{=} s^2-8Rr-2r^2$$

$\therefore$ via (1), (2), (3) and (4), $(**) \Leftrightarrow 2(s^2-8Rr-2r^2)-4Rr-r^2+9.\sqrt[3]{r^4s^2}$
$$\geq 4s.\sqrt[3]{r^2s} \Leftrightarrow \boxed{\left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)^3 \overset{(***)}{\geq} 64s^4r^2}$$

$$\text{Now, } \left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)^3$$
$$=(2s^2-20Rr-5r^2)^3+729r^4s^2$$
$$+27.\sqrt[3]{r^4s^2}.(2s^2-20Rr-5r^2)\left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)$$
$$\overset{\text{Mitrinovic}}{\geq} (2s^2-20Rr-5r^2)^3+729r^4s^2$$
$$+81r^2(2s^2-20Rr-5r^2)(2s^2-20Rr-5r^2+27r^2) \overset{?}{\geq} 64s^4r^2$$

$$\Leftrightarrow \boxed{\begin{array}{c}8s^6-(240Rr-200r^2)s^4+r^2s^2(2400R^2-5280Rr+3633r^2)-\\ r^3(8000R^3-26400R^2r+29040Rr^2+9035r^3)\underset{(****)}{\overset{?}{\geq}}0\end{array}} \textit{and}$$

$$\because 8(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0, \therefore \text{ in order to prove } (****), \text{it suffices to prove :}$$

$$\text{LHS of } (****)\geq 8(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (144Rr+80r^2)s^4-r^2s^2(3744R^2+1440Rr-3033r^2)$$

$$+r^3(24768R^3-4320R^2r-19440Rr^2-10035r^3)\overset{(*****)}{\geq}0 \textit{ and}$$

$$\because (144Rr+80r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0, \therefore \text{ in order to prove } (*****),$$

$$\text{it suffices to prove : LHS of } (*****)\geq(144Rr+80r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (864R^2-320Rr+2233r^2)s^2\overset{(******)}{\geq} r\begin{pmatrix}12096R^3+1760R^2r\\+10240Rr^2+12035r^3\end{pmatrix} \textit{ and finally,}$$

$$(864R^2-320Rr+2233r^2)s^2\overset{\text{Gerretsen}}{\geq}(864R^2-320Rr+2233r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(12096R^3+1760R^2r+10240Rr^2+12035r^3)$$

$$\Leftrightarrow 108t^3-700t^2+1693t-1450\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{\boxed{(t-2)(108t^2-484t+725)\overset{?}{\geq}0}}\rightarrow \text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2 \textit{ and discriminant of } (108t^2-484t+725)=484^2-432*725$$

$$=-78944\Rightarrow 108t^2-484t+725>0 \textit{ and so,}$$

$$(******)\Rightarrow(*****)\Rightarrow(****)\Rightarrow(***)\Rightarrow(**)\Rightarrow(*) \text{ is true}$$

$$\Rightarrow \frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2}\leq 1\ \forall\{x,y,z\}\in\mathbb{R}^+ \text{ such that : } xyz=1,$$

$$''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2}=\frac{x}{x^2+1+1}+\frac{y}{y^2+1+1}+\frac{z}{z^2+1+1}$$

$$\overset{AM-GM}{\underset{\smile}{\leq}}\frac{x}{2x+1}+\frac{y}{2y+1}+\frac{z}{2z+1}$$

$$=\left(\frac{1}{2}-\frac{1}{2(2x+1)}\right)+\left(\frac{1}{2}-\frac{1}{2(2y+1)}\right)+\left(\frac{1}{2}-\frac{1}{2(2z+1)}\right)$$

$$=\frac{3}{2}-\frac{1}{2}\left(\frac{yz}{2+yz}+\frac{zx}{2+zx}+\frac{xy}{2+xy}\right)\overset{CBS}{\underset{\smile}{\leq}}\frac{3}{2}-\frac{(\sqrt{yz}+\sqrt{zx}+\sqrt{xy})^2}{2(6+yz+zx+xy)}$$

$$=\frac{3}{2}-\frac{xy+yz+zx+2(\sqrt{x}+\sqrt{y}+\sqrt{z})}{2(6+xy+yz+zx)}\overset{AM-GM}{\leq}\frac{3}{2}-\frac{xy+yz+zx+2.3\sqrt[6]{xyz}}{2(6+xy+yz+zx)}=$$

$$=\frac{3}{2}-\frac{1}{2}=1$$

Equality holds iff $x=y=z=1$.

1423. If $x,y,z>0$, $x+y+z=3$ then:

$$\frac{xy}{\sqrt{z^2+8}}+\frac{xz}{\sqrt{y^2+8}}+\frac{yz}{\sqrt{x^2+8}}\leq 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{cases}\frac{xy}{\sqrt{z^2+8}}=\frac{xy}{\sqrt{z^2+1+7}}\underset{(2)}{\overset{(1)}{\leq}}\frac{\frac{1}{4}(x+y)^2}{\sqrt{2z+7}}=\frac{(x+y)^2}{4\sqrt{2z+7}}\\ \frac{xz}{\sqrt{y^2+8}}=\frac{xz}{\sqrt{y^2+1+7}}\underset{(2)}{\overset{(1)}{\leq}}\frac{\frac{1}{4}(x+z)^2}{\sqrt{2y+7}}=\frac{(x+z)^2}{4\sqrt{2y+7}}\\ \frac{yz}{\sqrt{x^2+8}}=\frac{yz}{\sqrt{x^2+1+7}}\underset{(2)}{\overset{(1)}{\leq}}\frac{\frac{1}{4}(y+z)^2}{\sqrt{2x+7}}=\frac{(y+z)^2}{4\sqrt{2x+7}}\end{cases}$$

Because, $x+y\geq 2\sqrt{xy} \Rightarrow xy\leq\frac{1}{4}(x+y)^2$ (1) $\quad z^2+1\geq 2z$ (2)

$$\frac{xy}{\sqrt{z^2+8}}+\frac{xz}{\sqrt{y^2+8}}+\frac{yz}{\sqrt{x^2+8}}\leq\frac{1}{4}\left(\frac{(x+y)^2}{\sqrt{2z+7}}+\frac{(x+z)^2}{\sqrt{2y+7}}+\frac{(y+z)^2}{\sqrt{2x+7}}\right)$$

$$\leq\frac{1}{4}\frac{\left((x+y)+(x+z)+(y+z)\right)^2}{3^{\frac{1}{2}}(2z+7+2y+7+2x+7)^{\frac{1}{2}}}=\frac{1}{4}\cdot\frac{6^2}{3^{\frac{1}{2}}(27)^{\frac{1}{2}}}=\frac{1}{4}\cdot\frac{36}{9}=1$$

Equality holds iff $x=y=z=1$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{yz}{\sqrt{x^2+8}}=\frac{3yz}{\sqrt{(1+8)(x^2+8)}}\leq\frac{3yz}{x+8}=\frac{3yz}{(z+x)+(x+y)+5}$$

$$\leq\frac{3yz}{(2+2+5)^2}\left(\frac{4}{z+x}+\frac{4}{x+y}+5\right)=\frac{4}{27}\left(\frac{yz}{z+x}+\frac{yz}{x+y}\right)+\frac{5yz}{27}.$$

Similarly, we have

$$\frac{xy}{\sqrt{z^2+8}} \le \frac{4}{27}\left(\frac{xy}{y+z}+\frac{xy}{z+x}\right)+\frac{5xy}{27} \text{ and } \frac{zx}{\sqrt{y^2+8}} \le \frac{4}{27}\left(\frac{zx}{x+y}+\frac{zx}{y+z}\right)+\frac{5zx}{27}.$$

Adding these inequalities, we obtain

$$\frac{xy}{\sqrt{z^2+8}}+\frac{yz}{\sqrt{x^2+8}}+\frac{zx}{\sqrt{y^2+8}} \le \frac{4}{27}\left(\frac{xy+zx}{y+z}+\frac{yz+xy}{z+x}+\frac{yz+zx}{x+y}\right)+\frac{5(xy+yz+zx)}{27}$$

$$\le \frac{4}{27}(x+y+z)+\frac{5(x+y+z)^2}{3.27}=\frac{4}{27}.3+\frac{5.3^2}{3.27}=1.$$

Equality holds iff $x=y=z=1$.

1424. Let $x,y,z \ge 0$ such that $x+y+z=3$. Prove that

$$\frac{1}{2x^2+2}+\frac{1}{2y^2+2}+\frac{1}{2z^2+2} \ge \frac{3}{4}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

For $x>0$, we have

$$\frac{1}{2x^2+2}=\frac{1}{2}-\frac{x^2}{2(x^2+1)} \overset{AM-GM}{\ge} \frac{1}{2}-\frac{x^2}{2.2x}=\frac{1}{2}-\frac{x}{4},$$

which is also true for $x=0$. Then

$$\frac{1}{2x^2+2}+\frac{1}{2y^2+2}+\frac{1}{2z^2+2} \ge \left(\frac{1}{2}-\frac{x}{4}\right)+\left(\frac{1}{2}-\frac{y}{4}\right)+\left(\frac{1}{2}-\frac{z}{4}\right)=\frac{3}{2}-\frac{x+y+z}{4}=$$

$=\frac{3}{2}-\frac{3}{4}=\frac{3}{4}$. Equality holds for: $x=y=z=1$.

1425. If $a,b,c \ge 0$ then:

$$9+2\sum_{cyc} a^5 \ge \sum_{cyc} a(a^2+3a+1)$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Daniel Sitaru-Romania

We will prove that:

$$2a^5 - a^3 - 3a^2 - a + 3 \geq 0, \forall a \geq 0$$

$$2a^5 - a^3 - 3a^2 - a + 3 =$$

$$= 2a^5 - 2a^4 + 2a^4 - 2a^3 + a^3 - a^2 - 2a^2 + 2a - 3a + 3 =$$

$$= 2a^4(a-1) + 2a^3(a-1) + a^2(a-1) - 2a(a-1) - 3(a-1) =$$

$$= (a-1)(2a^4 + 2a^3 + a^2 - 2a - 3) =$$

$$= (a-1)(2a^4 - 2a^3 + 4a^3 - 4a^2 + 5a^2 - 5a + 3a - 3) =$$

$$= (a-1)\left(2a^3(a-1) + 4a^2(a-1) + 5a(a-1) + 3(a-1)\right) =$$

$$= (a-1)^2(2a^3 + 4a^2 + 5a + 3) \geq 0$$

$$\sum_{cyc}\left(2a^5 - a^3 - 3a^2 - a + 3\right) \geq 0$$

$$2\sum_{cyc} a^5 + \sum_{cyc} 3 \geq \sum_{cyc}(a^3 + 3a^2 + a)$$

$$9 + 2\sum_{cyc} a^5 \geq \sum_{cyc} a(a^2 + 3a + 1)$$

Equality holds for: $a = b = c = 1$.

1426. If $x, y, z > 0, xyz = 1$ and $1 \leq \lambda \leq 2$, then :

$$\sum_{cyc} \frac{x}{x^2 + \lambda} \leq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{x}{x^2+\lambda} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{yz(x^2+\lambda)} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{x+\lambda yz} = \sum_{cyc} \frac{1}{x + yz + (\lambda-1)yz}$$

$$\overset{A-G}{\leq} \sum_{cyc} \frac{1}{2.\sqrt{xyz} + (\lambda-1)yz} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{2+ta} \begin{pmatrix} 0 \leq t = \lambda - 1 \leq 1 \text{ and} \\ yz = a, zx = b, xy = c \end{pmatrix}$$

$$\overset{?}{\leq}\frac{3}{\lambda+1}=\frac{3}{2+t}$$

$$\Leftrightarrow 3(2+ta)(2+tb)(2+tc)\overset{?}{\geq}(t+2)\left(\begin{matrix}(2+ta)(2+tb)+\\(2+tb)(2+tc)+(2+tc)(2+ta)\end{matrix}\right)$$

$$\overset{abc=1}{\Leftrightarrow}\boxed{t^3\left(3-\sum_{cyc}ab\right)+t^2\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)+t\left(4\sum_{cyc}a-12\right)\underset{(*)}{\overset{?}{\geq}}0}$$

$$\text{Now, }t^3\left(3-\sum_{cyc}ab\right)\overset{?}{\geq}t\left(3-\sum_{cyc}ab\right)\Leftrightarrow(t^3-t)\left(3-\sum_{cyc}ab\right)\overset{?}{\geq}0\to\text{true}$$

$$\because 0\leq t\leq 1\Rightarrow t^3-t\leq 0\text{ and }3-\sum_{cyc}ab\overset{A-G}{\leq}3-3.\sqrt[3]{a^2b^2c^2}\overset{abc=1}{=}0$$

$$\therefore t^3\left(3-\sum_{cyc}ab\right)\overset{(\bullet)}{\geq}t\left(3-\sum_{cyc}ab\right)$$

$$\text{Again, }t^3\left(3-\sum_{cyc}ab\right)\overset{?}{\geq}t^2\left(3-\sum_{cyc}ab\right)\Leftrightarrow(t^3-t^2)\left(3-\sum_{cyc}ab\right)\overset{?}{\geq}0\to\text{true}$$

$$\because 0\leq t\leq 1\Rightarrow t^3-t^2\leq 0\text{ and }3-\sum_{cyc}ab\overset{A-G}{\leq}3-3.\sqrt[3]{a^2b^2c^2}\overset{abc=1}{=}0$$

$$\therefore t^3\left(3-\sum_{cyc}ab\right)\overset{(\bullet\bullet)}{\geq}t^2\left(3-\sum_{cyc}ab\right)$$

$$\text{Also, }t^2\left(3-\sum_{cyc}a\right)\overset{?}{\geq}t\left(3-\sum_{cyc}a\right)\Leftrightarrow(t^2-t)\left(3-\sum_{cyc}a\right)\overset{?}{\geq}0\to\text{true}$$

$$\because 0\leq t\leq 1\Rightarrow t^2-t\leq 0\text{ and }3-\sum_{cyc}a\overset{A-G}{\leq}3-3.\sqrt[3]{abc}\overset{abc=1}{=}0$$

$$\therefore t^2\left(3-\sum_{cyc}a\right)\overset{(\bullet\bullet\bullet)}{\geq}t\left(3-\sum_{cyc}a\right)$$

$$\boxed{\text{Case 1}}\ \boxed{\sum_{cyc}a\geq\sum_{cyc}ab}\text{ and then : }t^2\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)\overset{?}{\geq}$$

$$t\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)\Leftrightarrow(t^2-t)\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)\overset{?}{\geq}0\to\text{true}$$

$$\because 0\leq t\leq 1\Rightarrow t^2-t\leq 0\text{ and }4\sum_{cyc}ab-4\sum_{cyc}a\leq 0$$

$$\therefore t^2\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)\overset{(\bullet\bullet\bullet\bullet)}{\geq}t\left(4\sum_{cyc}ab-4\sum_{cyc}a\right)\therefore(\bullet)+(\bullet\bullet\bullet\bullet)\Rightarrow\text{LHS of }(*)$$

$$\geq t\left(3-\sum_{cyc} ab\right)+t\left(4\sum_{cyc} ab-4\sum_{cyc} a\right)+t\left(4\sum_{cyc} a-12\right)$$

$$=t\left(3-\sum_{cyc} ab+4\sum_{cyc} ab-4\sum_{cyc} a+4\sum_{cyc} a-12\right)=3t\left(\sum_{cyc} ab-3\right)\geq 0$$

$$\because t\geq 0 \text{ and } \sum_{cyc} ab \overset{A-G}{\geq} 3.\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 3 \Rightarrow \sum_{cyc} ab-3\geq 0 \Rightarrow \boxed{(*)\text{ is true}}$$

$$\boxed{\text{Case 2}}\ \boxed{\sum_{cyc} ab\geq \sum_{cyc} a} \text{ and then : LHS of } (*) \overset{\text{via } (\bullet\bullet)}{\geq} t^2\left(3-\sum_{cyc} ab\right)$$

$$+t^2\left(4\sum_{cyc} ab-4\sum_{cyc} a\right)+t\left(4\sum_{cyc} a-12\right)$$

$$=t^2\left(3\sum_{cyc} ab-3\sum_{cyc} a\right)+t^2\left(3-\sum_{cyc} a\right)+t\left(4\sum_{cyc} a-12\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\geq} t^2\left(3\sum_{cyc} ab-3\sum_{cyc} a\right)+t\left(3-\sum_{cyc} a\right)+t\left(4\sum_{cyc} a-12\right)$$

$$=3t^2\left(\sum_{cyc} ab-\sum_{cyc} a\right)+3t\left(\sum_{cyc} a-3\right)\to \text{true}$$

$$\because \sum_{cyc} ab\geq \sum_{cyc} a \text{ and } \sum_{cyc} a \overset{A-G}{\geq} 3.\sqrt[3]{abc} \overset{abc=1}{=} 3 \Rightarrow \sum_{cyc} a-3\geq 0 \Rightarrow \boxed{(*)\text{ is true}}$$

$\therefore$ combining both cases, $(*)$ is true $\forall\, a,b,c>0\,|\,abc=1$ and $\forall\, t\in[0,1]$

$$\therefore \sum_{cyc}\frac{x}{x^2+\lambda}\leq \frac{3}{\lambda+1}\ \forall\, x,y,z>0\,|\,xyz=1 \text{ and } \forall\,\lambda\in[1,2],$$

$$"=" \text{ iff } x=y=z=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{x}{x^2+\lambda}=\sum_{cyc}\frac{x}{x^2+1+(\lambda-1)} \overset{AM-GM}{\leq} \sum_{cyc}\frac{x}{2x+\lambda-1}=\sum_{cyc}\left(\frac{1}{2}-\frac{\lambda-1}{2(2x+\lambda-1)}\right)$$

$$=\frac{3}{2}-\frac{\lambda-1}{2}\sum_{cyc}\frac{yz}{2+(\lambda-1)yz} \overset{CBS}{\leq} \frac{3}{2}-\frac{(\lambda-1)\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right)^2}{2[6+(\lambda-1)(yz+zx+xy)]}$$

$$=\frac{3}{2}-\frac{(\lambda-1)\left[xy+yz+zx+2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\right]}{2[6+(\lambda-1)(yz+zx+xy)]}\overset{AM-GM}{\leq}\frac{3}{2}$$
$$-\frac{(\lambda-1)[xy+yz+zx+2.3]}{2[6+(\lambda-1)(yz+zx+xy)]}$$

$$=1+\frac{3(2-\lambda)}{6+(\lambda-1)(yz+zx+xy)}\overset{AM-GM}{\leq}1+\frac{3(2-\lambda)}{6+(\lambda-1).3}=\frac{3}{\lambda+1},$$

as desired. Equality holds iff $x=y=z=1$.

1427. If $a,b,c>0$, then prove that

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^8\sum_{cyc}\left(\frac{a}{b}\right)^{16}\geq\left(\sum_{cyc}\frac{a}{b}\cdot\sum_{cyc}\frac{a}{c}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Daniel Sitaru-Romania

Lemma:

If $x,y,z\in\mathbb{R}$ then:

$$x^2+y^2+z^2\geq xy+yz+zx \qquad (1)$$

Proof:

$$(1)\Leftrightarrow 2x^2+2y^2+2z^2\geq 2xy+2yz+2zx\Leftrightarrow$$
$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0\Leftrightarrow$$
$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

Equality holds for $x=y=z$. Back to the problem:

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\overset{(1)}{\geq}\sum_{cyc}\frac{a}{b}\cdot\frac{b}{c}=\sum_{cyc}\frac{a}{c} \qquad (2)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^2\overset{(2)}{\geq}\sum_{cyc}\frac{a}{b} \qquad (3)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^8\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^4\overset{(3)}{\geq}\sum_{cyc}\frac{a}{c} \qquad (4)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^{16}\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^8\overset{(4)}{\geq}\sum_{cyc}\frac{a}{b}$$

By multypling $(1),(2),(3),(4)$:

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^8\cdot\sum_{cyc}\left(\frac{a}{b}\right)^{16}\geq\left(\sum_{cyc}\frac{a}{b}\cdot\sum_{cyc}\frac{a}{c}\right)^2$$

Equality holds for: $a = b = c$.

1428. **Prove that if $a, b, c > 0$, then:**

$$\sqrt{\sum(a+b)^2} \geq \left(\sqrt{\sum a^2} + \sqrt{3}\sum a\right) \cdot \frac{1}{2}$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$2\sqrt{\sum(a+b)^2} = \sqrt{(1+3)\left(\sum a^2 + \left(\sum a\right)^2\right)} \geq \sqrt{\sum a^2} + \sqrt{3}\sum a,$$

as desired. Equality holds iff $a = b = c$.

1429. **If $a + \mathrm{b} + \mathrm{c} = 3$, then :**

$$a^a + \mathrm{b}^\mathrm{b} + \mathrm{c}^\mathrm{c} \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x^x \overset{?}{\geq} x \Leftrightarrow x.\ln x \overset{?}{\geq} \ln x \Leftrightarrow (x-1).\ln x \overset{?}{\geq} 0 \to (1)$$

$\boxed{\text{Case 1}}$ $x \geq 1$ and then : $x - 1 \geq 0$ and $\ln x \geq 0 \Rightarrow (x-1).\ln x \geq 0$
$\Rightarrow (1)$ is true

$\boxed{\text{Case 2}}$ $x < 1$ and then : $x - 1 < 0$ and $\ln x < 0 \Rightarrow (x-1).\ln x > 0$
$\Rightarrow (1)$ is true

$\therefore$ combining both cases, (1) is always true $\therefore x^x \geq x$ and $\therefore a^a \geq a, \mathrm{b}^\mathrm{b} \geq \mathrm{b}, \mathrm{c}^\mathrm{c} \geq \mathrm{c}$

$$\therefore a^a + \mathrm{b}^\mathrm{b} + \mathrm{c}^\mathrm{c} \geq a + \mathrm{b} + \mathrm{c} = 3, '' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$

1430. **Let $a, \mathrm{b}, \mathrm{c} \in \mathbb{R}$. Prove that :**

$$3^{|a|} + 3^{|\mathrm{b}|} + 3^{|\mathrm{c}|} \geq 3 + \sqrt{a^2 + \mathrm{b}^2 + \mathrm{c}^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \mathrm{f}(x) = 3^x - x - 1\ \forall\, x \geq 0 \text{ and then : } \mathrm{f}'(x) = (\ln 3).3^x - 1 \to (1)$$

$$\text{Now}, \because x \ge 0 \text{ and } \ln 3 > 0, \therefore x.\ln 3 \ge \ln 1 \Rightarrow 3^x \ge 1 \text{ and } \therefore \text{via } (1),$$
$$f'(x) \ge \ln 3 - 1 > \ln e - 1 \Rightarrow f'(x) > 0 \therefore f(x) \text{ is } \uparrow \text{ on } [0,\infty) \Rightarrow f(x) \ge f(0) = 0$$
$$\forall x \in [0,\infty) \therefore 3^x - x - 1 \;\forall x \ge 0 \therefore 3^{|a|} \ge |a| + 1 \;\forall a \in \mathbb{R} \text{ and analogs}$$
$$\therefore 3^{|a|} + 3^{|b|} + 3^{|c|} \ge 3 + \sum_{cyc}|a| \overset{?}{\ge} 3 + \sqrt{a^2+b^2+c^2} \Leftrightarrow \sum_{cyc}|a| \overset{?}{\ge} \sqrt{a^2+b^2+c^2}$$
$$\Leftrightarrow \sum_{cyc} a^2 + 2\sum_{cyc}|a||b| \overset{?}{\ge} \sum_{cyc} a^2 \Leftrightarrow \sum_{cyc}|a||b| \overset{?}{\ge} 0 \to \text{true}$$
$$\therefore 3^{|a|} + 3^{|b|} + 3^{|c|} \ge 3 + \sqrt{a^2+b^2+c^2} \;\forall a,b,c \in \mathbb{R}, '' = '' \text{ iff } a = b = c = 0 \text{ (QED)}$$

1431.

If $a, b, c \in \mathbb{R}^+$ such that : $abc = 1$, then prove that :

$$\frac{1+2ab}{1+ac+bc} + \frac{1+2bc}{1+ac+ab} + \frac{1+2ac}{1+ab+bc} \ge 3$$

Proposed by Samed Ahmedov and Xumar Mammadli-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1+2ab}{1+ac+bc} + \frac{1+2bc}{1+ac+ab} + \frac{1+2ac}{1+ab+bc}$$
$$= \frac{1}{1+ac+bc} + \frac{1}{1+ac+ab} + \frac{1}{1+ab+bc} +$$
$$\frac{2a^2b^2}{ab+a^2bc+ab^2c} + \frac{2b^2c^2}{bc+abc^2+ab^2c} + \frac{2a^2c^2}{ac+a^2bc+abc^2}$$
$$\overset{\text{Bergstrom and } \because abc=1}{\ge} \frac{9}{3+2\sum_{cyc}ab} + \frac{2\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab + 2\sum_{cyc}a}$$
$$\ge \frac{9}{3+2\sum_{cyc}ab} + \frac{2\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab + \frac{2}{3}\left(\sum_{cyc}ab\right)^2}$$
$$\left(\because \left(\sum_{cyc}ab\right)^2 \ge 3abc\sum_{cyc}a \overset{abc=1}{=} 3\sum_{cyc}a \Rightarrow \sum_{cyc}a \le \frac{1}{3}\left(\sum_{cyc}ab\right)^2\right)$$
$$= \frac{9}{3+2\sum_{cyc}ab} + \frac{6\sum_{cyc}ab}{3+2\sum_{cyc}ab} = \frac{3\left(3+2\sum_{cyc}ab\right)}{3+2\sum_{cyc}ab} = 3$$
$$\therefore \frac{1+2ab}{1+ac+bc} + \frac{1+2bc}{1+ac+ab} + \frac{1+2ac}{1+ab+bc} \ge 3 \;\forall a,b,c \in \mathbb{R}^+ \,\middle|\, abc = 1,$$
$$'' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1+2ab}{1+bc+ca} + \frac{1+2bc}{1+ca+ab} + \frac{1+2ca}{1+ab+bc}$$

$$=\left(\frac{3+2(ab+bc+ca)}{1+bc+ca}-2\right)+\left(\frac{3+2(ab+bc+ca)}{1+ca+ab}-2\right)+\left(\frac{3+2(ab+bc+ca)}{1+ab+bc}-2\right)=$$

$$=[3+2(ab+bc+ca)]\left(\frac{1}{1+bc+ca}+\frac{1}{1+ca+ab}+\frac{1}{1+ab+bc}\right)-6$$

$$\overset{CBS}{\underset{(3)}{\geq}}[3+2(ab+bc+ca)].\frac{9}{3+2(ab+bc+ca)}-6=3,$$

as desired. Equality holds iff $a=b=c=1$.

1432. If $x,y,z>0\ \ x+y+z=3$ then:

$$\sum_{cyc}\frac{(\sqrt[3]{x}-\frac{1}{2}\sqrt[3]{y})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{y})^2}{x+y}\geq\frac{3}{2}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Cosghun Memmedov-Azerbaijan

$$\sum_{cyc}\frac{(\sqrt[3]{x}-\frac{1}{2}\sqrt[3]{y})^2+(\frac{\sqrt{3}}{2}\sqrt[3]{y})^2}{x+y}\geq\frac{3}{2}$$

$$let\ \sqrt[3]{x}=a\,,\sqrt[3]{y}=b\,,\sqrt[3]{z}=c$$

$$a^3+b^3+c^3=3\,,\ \frac{a^3+b^3+c^3}{3}\geq\left(\frac{a+b+c}{3}\right)^3\ \ =>a+b+c\leq 3$$

$$\sum_{cyc}\frac{(a-\frac{1}{2}b)^2+(\frac{\sqrt{3}}{2}b)^2}{a^3+b^3}$$

$$\geq\sum_{cyc}\frac{a^2-ab+b^2}{(a+b)(a^2-ab+b^2)}=\sum_{cyc}\frac{1}{a+b}\overset{Bergstrom}{\underset{(3)}{\geq}}\frac{9}{2(a+b+c)}\geq\frac{3}{2}$$

1433. If $a,b,c>0$ then:

$$a+b+c-\sqrt[3]{abc}\geq\frac{2}{3}\left(\sqrt{c}\cdot\sqrt[4]{ab}+\sqrt{a}\cdot\sqrt[4]{bc}+\sqrt{b}\cdot\sqrt[4]{ca}\right)$$

Proposed by Khaled Abd Imouti-Syria

Solution by Daniel Sitaru-Romania

$$a+b+c-\sqrt[3]{abc}\overset{AM-GM}{\underset{(3)}{\geq}}a+b+c-\frac{a+b+c}{3}=\frac{2}{3}(a+b+c)=$$

$$=\frac{2}{3}\left(\frac{a+b+c+c}{4}+\frac{b+c+a+a}{4}+\frac{c+a+b+b}{4}\right)\geq$$

$$\overset{AM-GM}{\geq}\frac{2}{3}\left(\sqrt[4]{a\cdot b\cdot c\cdot c}+\sqrt[4]{b\cdot c\cdot a\cdot a}+\sqrt[4]{c\cdot a\cdot b\cdot b}\right)=$$

$$=\frac{2}{3}\left(\sqrt{c}\cdot\sqrt[4]{ab}+\sqrt{a}\cdot\sqrt[4]{bc}+\sqrt{b}\cdot\sqrt[4]{ca}\right)$$

Equality holds for: $a=b=c$.

1434. If $a,b,c>0, a^2+b^2+c^2=3$ and $\lambda\geq 0$ then

$$\sum\frac{a^3}{a+\lambda}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

Solution by Amir Sofi – Kosovo

$$a,b,c>0,\lambda\geq 0, a^2+b^2+c^2=3$$

$$\frac{a^3}{a+\lambda}+\frac{b^3}{b+\lambda}+\frac{c^3}{c+\lambda}\geq\frac{3}{\lambda+1}$$

$$\frac{a^3}{a+\lambda}+\frac{b^3}{b+\lambda}+\frac{c^3}{c+\lambda}=\frac{a^4}{a^2+a\lambda}+\frac{b^4}{b^2+b\lambda}+\frac{c^4}{c^2+c\lambda}\geq$$

$$\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\lambda(a+b+c)}\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\lambda\sqrt{3(a^2+b^2+c^2)}}=$$

$$=\frac{3^2}{3+\lambda\sqrt{9}}=\frac{3}{\lambda+1}$$

Equality holds for $a=b=c=1$

1435. If $a,b,c>0, a+b+c=3$ and $\lambda\geq 0$ then:

$$\sum a\sqrt{2+\lambda b+\lambda c}\leq 3\sqrt{2(\lambda+1)}$$

Proposed by Marin Chirciu – Romania

Solution by Amir Sofi – Kosovo

$$a,b,c>0,\qquad a+b+c=3,\qquad \lambda\geq 0$$

$$a\sqrt{2+\lambda b+\lambda c}+b\sqrt{2+\lambda c+\lambda a}+c\sqrt{2+\lambda a+\lambda b}\leq 3\sqrt{2(\lambda+1)}$$

$$a\sqrt{2+\lambda b+\lambda c}+b\sqrt{2+\lambda c+\lambda a}+c\sqrt{2+\lambda a+\lambda b}=$$

$$=\sqrt{a}\sqrt{2a+\lambda a(3-a)}+\sqrt{b}\sqrt{2b+\lambda b(3-b)}+\sqrt{c}\sqrt{2c+\lambda c(3-c)}\le$$

$$\le\sqrt{a+b+c}\sqrt{2(a+b+c)+\lambda\big(3(a+b+c)-(a^2+b^2+c^2)\big)}=$$

$$=\sqrt{3}\sqrt{6+\lambda\big(9-(a^2+b^2+c^2)\big)}\le\sqrt{3}\sqrt{6+\lambda\left(9-\frac{(a+b+c)^2}{3}\right)}=3\sqrt{2(\lambda+1)}$$

Equality holds for $a=b=c=1$.

1436. If $a,b,c>0$, then prove that:

$$\prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^k}\right)\ge\left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^k}\right)\ge\left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n$$

$$\Leftrightarrow\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^4\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^8\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{16}\right)\cdots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}}\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}}\right)$$

$$\ge\left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n\to(*)$$

Firstly, we shall prove: $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}}\ge\sum_{cyc}\frac{a}{c}\ \forall m\in\mathbb{N}$ and

we shall prove via mathematical induction

$$\text{For } m=0,\ \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}}=\sum_{cyc}\left(\frac{a}{b}\right)^2\ge\sum_{cyc}\left(\frac{a}{b}\cdot\frac{b}{c}\right)=\sum_{cyc}\frac{a}{c}$$

$$\text{For } m=1,\ \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}}=\sum_{cyc}\left(\frac{a}{b}\right)^8\ge\sum_{cyc}\left(\left(\frac{a}{b}\right)^4\cdot\left(\frac{b}{c}\right)^4\right)=\sum_{cyc}\left(\frac{a}{c}\right)^4$$

$$\ge\sum_{cyc}\left(\left(\frac{a}{c}\right)^2\cdot\left(\frac{c}{b}\right)^2\right)=\sum_{cyc}\left(\frac{a}{b}\right)^2\ge\sum_{cyc}\left(\frac{a}{b}\cdot\frac{b}{c}\right)=\sum_{cyc}\frac{a}{c}$$

Let $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c}$ for some $m = k \in \mathbb{N} - \{0,1\} \to (1)$ and we shall prove :

$$\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c} \text{ for } m = k+1$$

$$\text{We have : } \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2(k+1)+1}} = \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k+1}.2^2} = \sum_{cyc}\left(\left(\frac{a}{b}\right)^{2^{2k+1}}\right)^4 \overset{\text{Holder}}{\ge}$$

$$\frac{1}{27}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k+1}}\right)^4 \overset{\text{via (1)}}{\ge} \frac{1}{27}\left(\sum_{cyc}\frac{a}{c}\right)^4 = \frac{1}{27}\left(\sum_{cyc}\frac{a}{c}\right)^3\left(\sum_{cyc}\frac{a}{c}\right) \overset{A-G}{\ge} \frac{1}{27}.3^3.\left(\sum_{cyc}\frac{a}{c}\right)$$

$$= \sum_{cyc}\frac{a}{c} \therefore \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c} \text{ for } m = k+1$$

$\therefore$ via the principle of mathematical induction, $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c}\ \forall\, m \in \mathbb{N}$

$$\Rightarrow \sum_{cyc}\left(\frac{a}{b}\right)^2, \sum_{cyc}\left(\frac{a}{b}\right)^8, \dots, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}} \ge \sum_{cyc}\frac{a}{c}$$

$$\Rightarrow \boxed{\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^8\right)\dots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}}\right) \ge \left(\sum_{cyc}\frac{a}{c}\right)^n} \to (\text{i})$$

Now, we shall prove : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} \ge \sum_{cyc}\frac{a}{b}\ \forall\, m \in \mathbb{N}^*$ and

we shall prove via mathematical induction

$$\text{For } m = 1, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} = \sum_{cyc}\left(\frac{a}{b}\right)^4 \ge \sum_{cyc}\left(\left(\frac{a}{b}\right)^2.\left(\frac{b}{c}\right)^2\right)$$

$$= \sum_{cyc}\left(\frac{a}{c}\right)^2 \ge \sum_{cyc}\left(\frac{a}{c}.\frac{c}{b}\right) = \sum_{cyc}\frac{a}{b}$$

$$\text{For } m = 2, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} = \sum_{cyc}\left(\frac{a}{b}\right)^{16} \ge \sum_{cyc}\left(\left(\frac{a}{b}\right)^8.\left(\frac{b}{c}\right)^8\right) = \sum_{cyc}\left(\frac{a}{c}\right)^8$$

$$\ge \sum_{cyc}\left(\left(\frac{a}{c}\right)^4.\left(\frac{c}{b}\right)^4\right) = \sum_{cyc}\left(\frac{a}{b}\right)^4 \ge \sum_{cyc}\left(\left(\frac{a}{b}\right)^2.\left(\frac{b}{c}\right)^2\right)$$

$$= \sum_{cyc}\left(\frac{a}{c}\right)^2 \ge \sum_{cyc}\left(\frac{a}{c}.\frac{c}{b}\right) = \sum_{cyc}\frac{a}{b}$$

Let $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} \ge \sum_{cyc}\frac{a}{b}$ for some $m = k \in \mathbb{N}^* - \{1,2\} \to (2)$

and we shall prove : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} \ge \sum_{cyc}\frac{a}{b}$ for $m = k+1$

$$\text{We have}: \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2(k+1)}} = \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k}.2^2} = \sum_{cyc}\left(\left(\frac{a}{b}\right)^{2^{2k}}\right)^4 \overset{\text{Holder}}{\geq} \frac{1}{27}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k}}\right)^4$$

$$\overset{\text{via }(2)}{\geq} \frac{1}{27}\left(\sum_{cyc}\frac{a}{b}\right)^4 = \frac{1}{27}\left(\sum_{cyc}\frac{a}{b}\right)^3\left(\sum_{cyc}\frac{a}{b}\right) \overset{A-G}{\geq} \frac{1}{27}.3^3.\left(\sum_{cyc}\frac{a}{b}\right) = \sum_{cyc}\frac{a}{b}$$

$$\therefore \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} \geq \sum_{cyc}\frac{a}{b} \text{ for } m = k+1$$

$$\therefore \text{via the principle of mathematical induction}, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}} \geq \sum_{cyc}\frac{a}{b}\ \forall m \in \mathbb{N}^*$$

$$\Rightarrow \sum_{cyc}\left(\frac{a}{b}\right)^4, \sum_{cyc}\left(\frac{a}{b}\right)^{16}, \dots, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}} \geq \sum_{cyc}\frac{a}{b}$$

$$\Rightarrow \boxed{\left(\sum_{cyc}\left(\frac{a}{b}\right)^4\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{16}\right)\dots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^n} \to (\text{ii})$$

$$\therefore (\text{i})\bullet(\text{ii}) \Rightarrow (*) \text{ is true} \therefore \prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^k}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n$$

$$\forall a,b,c > 0, '' = '' \text{ iff } a = b = c\ (\text{QED})$$

1437. If $x, y, z > 0$, *then prove that*:

$$3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx) \geq 4\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x^2+3\sum_{cyc}xy\right)^2$$

Proposed by Mihaly Bencze,Neculai Stanciu-Romania
Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a = 2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc}x = \sum_{cyc}a = 2s \Rightarrow \sum_{cyc}x \overset{(\bullet)}{=} s \Rightarrow x = s-a, y = s-b, z = s-c$$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2s \text{ and}, \sum_{cyc}xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc}xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{Now}, 3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx)$$

$$= 3\prod_{cyc}(3y(y+x+z)+x^2+z^2+zx) \geq 3\prod_{cyc}\left(3y(y+x+z)+\frac{3}{4}(x+z)^2\right)$$

$$= 81\prod_{cyc}\left(s(s-b)+\frac{b^2}{4}\right) = \frac{81}{64}\prod_{cyc}(b^2-4sb+4s^2) = \frac{81}{64}\prod_{cyc}(2s-b)^2$$

$$= \frac{81}{64}\left(\prod_{cyc}(b+c)\right)^2 = \frac{81}{64}.4s^2(s^2+2Rr+r^2)^2$$

$$= \frac{81s^2(s^2+2Rr+r^2)^2}{16} \overset{?}{\geq} 4\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x^2+3\sum_{cyc}xy\right)^2$$

$$= 4\left(\sum_{cyc}x\right)^2\left(\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)+\sum_{cyc}xy\right)^2$$

$$= 4\left(\sum_{cyc}x\right)^2\left(\left(\sum_{cyc}x\right)^2+\sum_{cyc}xy\right)^2 \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} 4s^2(s^2+4Rr+r^2)^2$$

$$\Leftrightarrow \frac{9(s^2+2Rr+r^2)}{4} \overset{?}{\geq} 2(s^2+4Rr+r^2) \Leftrightarrow s^2-14Rr+r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2-16Rr+5r^2+2r(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because s^2-16Rr+5r^2 \overset{\text{Gerretsen}}{\geq} 0 \text{ and } 2r(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore 3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx) \geq 4\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x^2+3\sum_{cyc}xy\right)^2$$

$$\forall\, x,y,z>0, '' ='' \text{ iff } x=y=z \text{ (QED)}$$

1438. If $a,b>0$ then:

$$a^b b^a \leq \left(\frac{a+b}{2}\right)^{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Pham Duc Nam-Vietnam

$$a_i \geq 0 (i=1,2,\dots,n)$$

$$x_i > 0 (i=1,2,\dots,n), \sum_{i=1}^{n} x_i = 1$$

$$\textstyle\sum_{i=1}^{n} x_i a_i \geq \prod_{i=1}^{n} a_i^{x_i} \qquad (1)$$

Let: $a_1 = a, a_2 = b, x_1 = \frac{b}{a+b}, x_2 = \frac{a}{a+b}, x_1 + x_2 = 1$

$$\Rightarrow a^{\frac{b}{a+b}}b^{\frac{a}{a+b}} \le a\left(\frac{b}{a+b}\right) + b\left(\frac{a}{a+b}\right) = \frac{2ab}{a+b}$$

But: $\frac{2ab}{a+b} \le \frac{a+b}{2} \because \frac{2ab}{a+b} \le \frac{a+b}{2} \Leftrightarrow (a+b)^2 \ge 4ab \Leftrightarrow 4ab \Leftrightarrow (a-b)^2 \ge 0$

which is true for all $a, b > 0$

$$\Rightarrow a^{\frac{b}{a+b}}b^{\frac{a}{a+b}} < \frac{a+b}{2} \Leftrightarrow \left(a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}\right)^{a+b} \le \left(\frac{a+b}{2}\right)^{a+b} \Leftrightarrow a^b b^a \le \left(\frac{a+b}{2}\right)^{a+b}$$

Equality holds if and only if $a = b$.

* Prove (1)

$$\sum_{i=1}^{n} x_i a_i = a = a^{\sum_{i=1}^{n} x_i} = \prod_{i=1}^{n} a^{x_i} \Rightarrow (1) \Leftrightarrow \prod_{i=1}^{n} a^{x_i} \ge \prod_{i=1}^{n} a_i^{x_i} \Leftrightarrow \prod_{i=1}^{n}\left(\frac{a_i}{a}\right)^{x_i} \le 1$$

$$x \le e^{x-1} \forall x \Rightarrow \frac{a_i}{a} \le e^{\frac{a_i}{a}-1} \Rightarrow \left(\frac{a_i}{a}\right)^{x_i} \le e^{\frac{x_i a_i}{a} - x_i} \Rightarrow$$

$$\Rightarrow \prod_{i=1}^{n}\left(\frac{a_i}{a}\right)^{x_i} \le \prod_{i=1}^{n} e^{\frac{x_i a_i}{a} - x_i} \Leftrightarrow \prod_{i=1}^{n}\left(\frac{a_i}{a}\right)^{x_i} \le e^{\sum_{i=1}^{n}\left(\frac{x_i a_i}{a} - x_i\right)} = e^{1-1} = 1 \Rightarrow$$

$\Rightarrow$ (1) is true.

1439. If $x, y, z > 0$ then:

$$\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} + 27(x^3 + y^3 + z^3) \ge 12$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Ravi Prakash-New Delhi-India

For $t > 0, \frac{1}{t} + 27t^3 = \frac{1}{3t} + \frac{1}{3t} + \frac{1}{3t} + 27t^3 \ge 4\left(\frac{1}{3^3 t^3} \cdot 27t^3\right)^{\frac{1}{4}} = 4$

Equality when $t = \frac{1}{3}$. For $x, y, z > 0$

$$\frac{1}{2}\left(\frac{x}{yz} + \frac{y}{zx}\right) + 27z^3 \ge \sqrt{\frac{x}{yz} \cdot \frac{y}{zx}} + 27z^3$$

$$\ge \frac{1}{z} + 27z^3 \ge 4 \quad (1)$$

Similarly, $\frac{1}{2}\left(\frac{x}{yz} + \frac{z}{xy}\right) + 27y^3 \ge 4 \quad (2)$

and $\frac{1}{2}\left(\frac{y}{zx} + \frac{z}{xy}\right) + 2yx^3 \ge 12 \quad (3)$

Adding (1), (2) and (3), we get the desired inequality.

1440. If $a+b\neq 0$, then :

$$a^2+b^2+\left(\frac{1-ab}{a+b}\right)^2\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z\in\mathbb{R},\sum_{cyc}(x-y)^2\geq 0\Rightarrow 2\sum_{cyc}x^2+\sum_{cyc}x^2\geq\sum_{cyc}x^2+2\sum_{cyc}xy$$

$$\Rightarrow\sum_{cyc}x^2\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2\rightarrow(1)\text{ and also},\forall\, x,y\in\mathbb{R},$$

$$(x+y)^2-4xy\geq 0\Rightarrow(x+y)^2\geq 4xy\rightarrow(2)$$

$$\therefore a^2+b^2+\left(\frac{1-ab}{a+b}\right)^2\overset{\text{via }(1)}{\geq}\frac{1}{3}\left(a+b+\frac{1-ab}{a+b}\right)^2=\frac{1}{3}\cdot\frac{\left((a+b)^2-ab+1\right)^2}{(a+b)^2}$$

$$=\frac{1}{3}\cdot\frac{\left(a^2+b^2+ab+1\right)^2}{(a+b)^2}\geq\frac{1}{3}\cdot\frac{\left(\frac{3}{4}(a+b)^2+1\right)^2}{(a+b)^2}$$

$$\left(\begin{array}{c}\because 4\left(a^2+b^2+ab\right)-3(a+b)^2=(a-b)^2\geq 0\ \forall\, a,b\in\mathbb{R}\\ \Rightarrow a^2+b^2+ab\geq\frac{3}{4}(a+b)^2\ \forall\, a,b\in\mathbb{R}\end{array}\right)\overset{\text{via }(2)}{\geq}\frac{1}{3}\cdot\frac{4.\frac{3}{4}(a+b)^2.1}{(a+b)^2}$$

$$=1\therefore a^2+b^2+\left(\frac{1-ab}{a+b}\right)^2\geq 1\ \forall\, a,b\in\mathbb{R}\,|\,a+b\neq 0,$$

$$''='' \text{ iff }\left(a=b=\frac{1}{\sqrt{3}}\right)\text{ or }\left(a=b=-\frac{1}{\sqrt{3}}\right)\text{ (QED)}$$

1441. If $abc=1$, then :

$$\sqrt{\frac{2}{1+a^2}}+\sqrt{\frac{2}{1+b^2}}+\sqrt{\frac{2}{1+c^2}}\leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } abc=1,\text{ we may assign } a=\frac{yz}{x^2},b=\frac{zx}{y^2},c=\frac{xy}{z^2}\text{ and then :}$$

$$\sum_{cyc}\frac{1}{\sqrt{1+a^2}}=\sum_{cyc}\frac{1}{\sqrt{1+\frac{y^2z^2}{x^4}}}=\sum_{cyc}\frac{x^2}{\sqrt{x^4+y^2z^2}}$$

$$=\sum_{cyc}\frac{x^2.\sqrt{y^4+z^2x^2}.\sqrt{z^4+x^2y^2}}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}$$

$$=\sum_{cyc}\frac{\left(x.\sqrt{y^4+z^2x^2}\right)\left(x.\sqrt{z^4+x^2y^2}\right)}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}\overset{CBS}{\le}$$

$$\frac{1}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}.\left(\begin{matrix}\left(\sqrt{x^2(y^4+z^2x^2)+y^2(z^4+x^2y^2)+z^2(x^4+y^2z^2)}\right)\\ *\\ \left(\sqrt{x^2(z^4+x^2y^2)+y^2(x^4+y^2z^2)+z^2(y^4+z^2x^2)}\right)\end{matrix}\right)$$

$$\overset{?}{\le}\frac{3}{\sqrt{2}}$$

$$\Leftrightarrow 9\left(x^4+y^2z^2\right)\left(y^4+z^2x^2\right)\left(z^4+x^2y^2\right)$$

$$-2\left(\begin{matrix}\left(x^2\left(y^4+z^2x^2\right)+y^2\left(z^4+x^2y^2\right)+z^2\left(x^4+y^2z^2\right)\right)\\ *\\ \left(x^2\left(z^4+x^2y^2\right)+y^2\left(x^4+y^2z^2\right)+z^2\left(y^4+z^2x^2\right)\right)\end{matrix}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc}x^6\right)+\sum_{cyc}x^6y^6-6x^4y^4z^4\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(3x^2y^2z^2+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right)\right)$$

$$+\left(3x^4y^4z^4+\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^4y^4-x^2y^2z^2\sum_{cyc}x^2\right)\right)-6x^4y^4z^4\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right)$$

$$+\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^4y^4-x^2y^2z^2\sum_{cyc}x^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc}x^2\right).\frac{1}{2}\sum_{cyc}\left(x^2-y^2\right)^2+\left(\sum_{cyc}x^2y^2\right).\frac{1}{2}\sum_{cyc}\left(x^2y^2-y^2z^2\right)^2\overset{?}{\ge}0$$

$$\to \text{true}\ \therefore\ \sqrt{\frac{2}{1+a^2}}+\sqrt{\frac{2}{1+b^2}}+\sqrt{\frac{2}{1+c^2}}\le 3\ \forall\, a,b,c\in\mathbb{R}\,\middle|\,abc=1,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1442. If $a,b,c>0$ and $a^2+b^2+c^2=3$, then prove that :

$$3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 15$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s \Rightarrow \sum_{cyc}a=s \to (1) \Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc}ab=4Rr+r^2 \to (3), \sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab$$

$$\overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc}a^2=s^2-8Rr-2r^2 \to (4)$$

$$\text{Now, } 3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 15 \overset{\because a^2+b^2+c^2=3}{\Leftrightarrow}$$

$$3\sum_{cyc}a+\frac{2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)}{3abc}\ge 5.\sqrt{3\sum_{cyc}a^2}$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow} \left(3s+\frac{2(4Rr+r^2)(s^2-8Rr-2r^2)}{3r^2s}\right)^2\ge 75(s^2-8Rr-2r^2)$$

$$\Leftrightarrow (32R^2+88Rr-277r^2)s^4-rs^2(512R^3+960R^2r-2316Rr^2-631r^3)$$
$$+8r^2(4R+r)^4 \overset{(*)}{\ge} 0$$

$$\because 32R^2+88Rr-277r^2=(R-2r)(32R+152r)+27r^2 \overset{\text{Euler}}{\ge} 27r^2>0$$

$$\therefore \text{LHS of } (*) \overset{\text{Gerretsen}}{\ge} (32R^2+88Rr-277r^2)(16Rr-5r^2)s^2$$
$$-rs^2(512R^3+960R^2r-2316Rr^2-631r^3)+8r^2(4R+r)^4 \overset{?}{\ge} 0$$
$$\Leftrightarrow 9(8R-7r)(R-8r)s^2+2(4R+r)^4 \underbrace{\overset{?}{\ge}}_{(**)} 0$$

$\boxed{Case\ 1}$ $R-8r\ge 0$ and then : LHS of $(**)\ge 2(4R+r)^4>0 \Rightarrow (**)$ is true (strict inequality)

$\boxed{Case\ 2}$ $R-8r<0$ and then : LHS of $(**)=-9(8R-7r)(8r-R)s^2+$
$$2(4R+r)^4 \overset{\text{Gerretsen}}{\ge} -9(8R-7r)(8r-R)(4R^2+4Rr+3r^2)+2(4R+r)^4 \overset{?}{\ge} 0$$
$$\Leftrightarrow 800t^4-1756t^3-132t^2+131t+1514 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(800t^2+1444t+2444)+4131\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$
$\Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true $\forall \Delta ABC$

$$\therefore 3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 15\ \forall\, a,b,c>0 \,\middle|\, a^2+b^2+c^2=3,$$
$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $3a+\frac{2}{a}=\frac{a^2+9}{2}+\frac{(4-a)(a-1)^2}{2a}\ge\frac{a^2+9}{2}$, for all $a<2$.

Then:

$$3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\left(3a+\frac{2}{a}\right)+\left(3b+\frac{2}{b}\right)+\left(3c+\frac{2}{c}\right)$$
$$\ge\frac{a^2+b^2+c^2+3.9}{2}=15,$$

as desired. Equality holds iff $a=b=c=1$.

1443. Let $\{x,y,z\}$ be positive real nmbers such that : $x+y+z=3$. Prove that :

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\le\frac{3\sqrt{2}}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}=\frac{1}{\sqrt{(x+y)(y+z)(z+x)}}\cdot\sum_{cyc}\left(\sqrt{x(y+z)}.\sqrt{z+x}\right)\overset{CBS}{\le}$$

$$\frac{1}{\sqrt{(x+y)(y+z)(z+x)}}\cdot\sqrt{\sum_{cyc}x(y+z)}\sqrt{\sum_{cyc}(z+x)}=\frac{\sqrt{2\sum_{cyc}xy}.\sqrt{2\sum_{cyc}x}}{\sqrt{(\sum_{cyc}x)(\sum_{cyc}xy)-xyz}}$$

$$\overset{?}{\le}\frac{3}{\sqrt{2}}\Leftrightarrow 9\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\overset{?}{\ge}8\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\ge}9xyz\to\text{true}\because\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{A-G}{\ge}3.\sqrt[3]{xyz}.3.\sqrt[3]{x^2y^2z^2}$$

$$= 9xyz \therefore \sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2}\ \forall\, x,y,z>0,'' ='' \text{ iff } x=y=z\ (\text{QED})$$

Solution 2 by Pham Duc Nam-Vietnam

$x,y,z>0\ \ x+y+z=3$

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2}\ (1)?\quad *(1) <=> \sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} \le 3$$

We will prove this inequality is true for any $x,y,z>0$

$$*\ \sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} = \sqrt{\frac{2x(x+z)}{(x+y)(x+z)}}+\sqrt{\frac{2y(y+x)}{(y+z)(y+x)}}+\sqrt{\frac{2z(z+y)}{(x+z)(y+z)}}$$

$$\le \sqrt{(2x+2y+2z)\left(\frac{2x}{(x+y)(x+z)}+\frac{2y}{(y+z)(y+x)}+\frac{2z}{(x+z)(y+z)}\right)}$$

So, we only need to prove:

$$(2x+2y+2z)\left(\frac{2x}{(x+y)(x+z)}+\frac{2y}{(y+z)(y+x)}+\frac{2z}{(x+z)(y+z)}\right) \le$$

$$<=> 4(x+y+z)\big(x(y+z)+y(x+z)+z(x+y)\big) \le 9(x+y)(y+z)(x+z)$$

$$<=> 8(x+y+z)(xy+yz+xz) \le 9(x+y)(y+z)(x+z)$$

$$<=> (x+y)(y+z)(x+z) \ge \frac{8}{9}(x+y+z)(xy+yz+xz) \ge \frac{8}{9}\left(\sqrt[3]{xyz}\right)\left(\sqrt[3]{x^2y^2z^2}\right) = 8xyz$$

which is true by AM − GM

$$<=> \sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} \le 3\ \forall\, x,y,z>0$$

$$<=> \sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2}, \textit{equality holds if and only } x=y=z$$

In this case: $x+y+z=3 \Rightarrow$ *equality holds if and only* $x=y=z=1$

1444. If $a,b,c>0$ and $a^5+b^5+c^5=3$, then prove that:

$$\sqrt[3]{a^2(a^5+a^3b^2+b^5)}+\sqrt[3]{b^2(b^5+b^3c^2+c^5)}+\sqrt[3]{c^2(c^5+c^3a^2+a^5)} \le 3\sqrt[3]{a^2+b^2+c^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Holder,}\left(\sum_{cyc} x\right)^3 \le 9\sum_{cyc} x^3\ \forall\, x,y,z>0 \Rightarrow \sum_{cyc} x \le \sqrt[3]{9\sum_{cyc} x^3}$$

$$\therefore \sum_{cyc}\sqrt[3]{a^2(a^5+a^3b^2+b^5)} \le \sqrt[3]{9\sum_{cyc} a^2(a^5+a^3b^2+b^5)} \overset{a^5+b^5+c^5=3}{=}$$

$$\sqrt[3]{9\sum_{cyc} a^2(3-c^5+a^3b^2)} = \sqrt[3]{27\sum_{cyc} a^2 - 9\sum_{cyc} c^5a^2 + 9\sum_{cyc} a^5b^2}$$

$$= \sqrt[3]{27\sum_{cyc} a^2 - 9\sum_{cyc} a^5b^2 + 9\sum_{cyc} a^5b^2} = 3\sqrt[3]{a^2+b^2+c^2},$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1445. If $a,b,c>0, a^5+b^5+c^5=3$ then prove that:

$$\frac{a\sqrt[3]{a(a^5+a^3b^2+b^5)^5}}{\sqrt[3]{(a^5+a^2b^3+b^5)^2}} + \frac{b\sqrt[3]{b(b^5+b^3c^2+c^5)^5}}{\sqrt[3]{(b^5+b^2c^3+c^5)^2}} + \frac{c\sqrt[3]{c(c^5+c^3a^2+a^5)^5}}{\sqrt[3]{(c^5+c^2a^3+a^5)^2}}$$
$$\ge \sqrt{3(a^2+b^2+c^2)^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc}\frac{a\sqrt[3]{a(a^5+a^3b^2+b^5)^5}}{\sqrt[3]{(a^5+a^2b^3+b^5)^2}} = \sum_{cyc}\frac{(a^7+a^5b^2+a^2b^5)^{\frac{5}{3}}}{(a^8+a^5b^3+a^3b^5)^{\frac{2}{3}}} \ge \frac{\left(\sum_{cyc}(a^7+a^5b^2+a^2b^5)\right)^{\frac{5}{3}}}{\left(\sum_{cyc}(a^8+a^5b^3+a^3b^5)\right)^{\frac{2}{3}}}$$

$$= \sqrt[3]{\frac{\left((a^2+b^2+c^2)(a^5+b^5+c^5)\right)^5}{\left((a^3+b^3+c^3)(a^5+b^5+c^5)\right)^2}} = \sqrt[3]{\frac{27(a^2+b^2+c^2)^5}{(a^3+b^3+c^3)^2}} \ge \sqrt{3(a^2+b^2+c^2)^5},$$

the last inequality is true by Power Mean Inequality:

$$a^3+b^3+c^3 \le 3\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^3} = 3.$$

Equality holds iff $a=b=c$.

1446. If $a,b,c>0, abc=1$ then:

$$\frac{a^6+b^6}{c^4(a^5+b^5)} + \frac{b^6+c^6}{a^4(b^5+c^5)} + \frac{c^6+a^6}{b^4(c^5+a^5)} \ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{a^6+b^6}{c^4(a^5+b^5)}+\frac{b^6+c^6}{a^4(b^5+c^5)}+\frac{c^6+a^6}{b^4(c^5+a^5)}=\sum_{cyc}\frac{a^6+b^6}{c^4(a^5+b^5)}\ge$$

$$\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^5+b^5}{c^4(a^4+b^4)}\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^4+b^4}{c^4(a^3+b^3)}\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^3+b^3}{c^4(a^2+b^2)}\ge$$

$$\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^2+b^2}{c^4(a^1+b^1)}\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^1+b^1}{c^4(a^0+b^0)}\ge\frac{1}{2}\sum_{cyc}\left(\frac{a}{c^4}+\frac{b}{c^4}\right)\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2}\cdot 6\sqrt[6]{\frac{a}{c^4}\cdot\frac{b}{c^4}\cdot\frac{b}{a^4}\cdot\frac{c}{a^4}\cdot\frac{c}{b^4}\cdot\frac{a}{b^4}}=3\sqrt[6]{\frac{1}{(abc)^6}}=3\sqrt[6]{\frac{1}{1^6}}=3$$

Equality holds for $a=b=c=1$.

1447. **If $a,b,c>0, a+b+c=3$ and $n,k\in\mathbb{N}$ then:**

$$\sum\frac{a^{n+k}}{\sqrt{b+c}}\ge\frac{a^k+b^k+c^k}{\sqrt{2}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\sum\frac{a^{m+k}}{\sqrt{b+c}}\underset{CBS}{\ge}\frac{\sum a^m\cdot\sum a^k}{3}$$

$$\sum\frac{a^{m+k}}{\sqrt{b+c}}\underset{Chebysev}{\ge}\frac{1}{3}\cdot\sum a^{m+k}\cdot\sum\frac{1}{\sqrt{b+c}}\underset{CBS}{\ge}\frac{1}{3}\cdot\frac{\sum a^m\cdot\sum a^k}{3}\cdot\frac{(1+1+1)^2}{\sqrt{6(a+b+c)}}\ge$$

$$\underset{CBS}{\ge}\frac{1}{9}\cdot\sum a^k\cdot\frac{1}{3^{m-1}}\cdot\left(\sum a\right)^m\cdot\frac{9}{\sqrt{6\times 3}}=\frac{1}{9}\cdot\sum a^k\cdot\frac{1}{3^{m-1}}\cdot(3)^m\cdot\frac{9}{3\sqrt{2}}$$

$$=\frac{1}{9}\cdot\sum a^k\cdot\frac{3\times 9}{3\sqrt{2}}=\frac{\sum a^k}{\sqrt{2}}$$

Solution 2 by Eric Cismaru-Romania

Without loss of generality, let us assume $a\ge b\ge c$.

This implies that $a^k\ge b^k\ge c^k$ and that $\frac{a^n}{\sqrt{b+c}}\ge\frac{b^n}{\sqrt{a+c}}\ge\frac{c^k}{\sqrt{a+b}}$. Applying now

Chebysev's Inequality for the sequences $\left\{\frac{a^n}{\sqrt{b+c}},\frac{b^n}{\sqrt{a+c}},\frac{c^n}{\sqrt{b+a}}\right\}$ and $\{a^k,b^k,c^k\}$, we obtain

$$\sum \frac{a^{n+k}}{\sqrt{b+c}} \geq \frac{1}{3}\left(\sum a^k\right)\left(\sum \frac{a^n}{\sqrt{b+c}}\right)$$

Using Holder's Inequality

$$\left(\sum \frac{a^n}{\sqrt{b+c}}\right)^{\frac{1}{n}} \cdot \left(\sum \sqrt{b+c}\right)^{\frac{1}{n}} \underbrace{(1+1+1)^{\frac{1}{n}}(1+1+1)^{\frac{1}{n}} \dots (1+1+1)^{\frac{1}{n}}}_{n-2\ times} \geq \sum a$$

and by raising this relationship to the power of n and dividing by 3^{n-2}, we find that

$$\sum \frac{a^n}{\sqrt{b+c}} \geq \frac{(\sum a)^n}{(\sum \sqrt{b+c}) \cdot 3^{n-2}} = \frac{3^n}{3^{n-2} \cdot (\sum \sqrt{b+c})}$$

But $\sum \sqrt{b+c} \overset{C.B.S}{\leq} \sqrt{6(a+b+c)} = 3\sqrt{2} \Leftrightarrow \sum \frac{a^n}{\sqrt{b+c}} \geq \frac{3^n}{3^{n-1}.\sqrt{2}} = \frac{3}{\sqrt{2}}$, which is equivalent to

$$\sum \frac{a^{n+k}}{\sqrt{b+c}} \geq \left(\sum a^k\right) \cdot \frac{1}{3} \cdot \frac{3}{\sqrt{2}} = \frac{a^k+b^k+c^k}{\sqrt{2}}$$

In conclusion, $\sum \frac{a^{n+k}}{\sqrt{b+c}} \geq \frac{a^k+b^k+c^k}{\sqrt{2}}$

Equality holds when $a = b = c = 1$.

1448.

If $x_k > 0$ $(k = 1, 2, \dots, n)$, then prove that:

$$\sum_{cyc} \frac{(x_1+x_2+x_3)^5 - x_1^5 - x_2^5 - x_3^5}{(x_1+x_2+x_3)^3 - x_1^3 - x_2^3 - x_3^3} \leq 10 \sum_{k=1}^{n} x_k$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(x_1+x_2+x_3)^5 - x_1^5 - x_2^5 - x_3^5}{(x_1+x_2+x_3)^3 - x_1^3 - x_2^3 - x_3^3} =$$

$$= \frac{5(x_1+x_2)(x_2+x_3)(x_3+x_1)(x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1)}{3(x_1+x_2)(x_2+x_3)(x_3+x_1)}$$

$$= \frac{5}{3}(x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1) \leq \frac{10}{3}(x_1^2+x_2^2+x_3^2).$$

Therefore

$$\sum_{cyc}\frac{(x_1+x_2+x_3)^5-{x_1}^5-{x_2}^5-{x_3}^5}{(x_1+x_2+x_3)^3-{x_1}^3-{x_2}^3-{x_3}^3}\le\sum_{cyc}\frac{10}{3}({x_1}^2+{x_2}^2+{x_3}^2)=10\sum_{k=1}^{n}{x_k}^2.$$

Equality holds iff $x_1=x_2=\cdots=x_n$.

1449. If $a,b,c>0$, then prove that:

$$\sum\frac{(a+b)(a^2+b^2)}{4c}\ge\sum a^2$$

Proposed by Neculai Stanciu-Romania

Solution by Eric Cismaru-Romania

$$\sum\frac{(a+b)(a^2+b^2)}{4c}\overset{AM-GM}{\ge}\frac{2ab(a+b)}{4c}=\sum\frac{ab(a+b)}{2c}\ge\sum a^2$$

so it is sufficient to prove that $\sum\frac{ab(a+b)}{c}\ge 2\cdot\left(\sum a^2\right)$

Grouping the terms from the left – hand side and using the AM-GM inequality, we obtain

$$\left(\frac{a^2b}{c}+\frac{a^2c}{b}\right)+\left(\frac{b^2c}{a}+\frac{b^2a}{c}\right)+\left(\frac{c^2b}{a}+\frac{c^2a}{b}\right)\ge\sum 2a^2$$

which is exactly what we wanted to prove.

Equality holds when $ab=ac\Leftrightarrow b=c$ and when $bc=ab\Leftrightarrow a=c\Rightarrow a=b=c$.

1450. If $a_k>0$ $(k=1,2,\dots,n)$, $\lambda\ge 2n+1$ and $\sum_{k=1}^{n}a_k=n$, then prove that

$$\sum_{k=1}^{n}\frac{1}{\lambda+a_k^2}\le\frac{n}{1+\lambda}$$

What happens if λ does not verify the hypothesis?

Proposed by Neculai Stanciu – Romania

Solution by Eric Cismaru – Romania

Let us show that $\frac{1}{\lambda+a_k^2}\le\frac{\lambda+3-2a_k}{(\lambda+1)^2}$, for any $k=\overline{1,n}$.

This inequality is equivalent to

$$(\lambda+3-2a_k)(\lambda+a_k^2)\ge(\lambda+1)^2\Leftrightarrow$$

$$\lambda^2+\lambda a_k^2+3\lambda+3a_k^2-\lambda 2a_k-2a_k^3\geq\lambda^2+2\lambda+1$$

$\Leftrightarrow 2a_k^3+\lambda 2a_k-3a_k^2-\lambda-\lambda a_k^2+1\leq 0\Leftrightarrow(a_k-1)^2(2a_k+1-\lambda)\leq 0$, with $2a_k+1\leq 2n+1\leq\lambda\Leftrightarrow a_k\leq n$, which is true because all the terms add up to n and are all positive reals.

Thus, we have shown that $\frac{1}{\lambda+a_k^2}\leq\frac{\lambda+3-2a_k}{(\lambda+1)^2}$, for any $k=\overline{1,n}$.

Using this result, we have

$$\sum_{k=1}^{n}\frac{1}{\lambda+a_k^2}\leq\sum_{k=1}^{n}\frac{\lambda+3-2a_k}{(\lambda+1)^2}=\frac{n\lambda+3n-2n}{(\lambda+1)^2}=\frac{n(\lambda+1)}{(\lambda+1)^2}=\frac{n}{\lambda+1}$$

Equality holds when $a_k=1, k=\overline{1,n}$.

1451. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$(a^3+a)^b(b^3+b)^c(c^3+c)^a\leq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)\text{ and}$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\sum_{cyc}a\overset{\text{via (1),(2) and (3)}}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)\to(5)$$

$$\text{Now, }\sqrt[a+b+c]{(a^3+a)^b(b^3+b)^c(c^3+c)^a}$$
$$=\sqrt[a+b+c]{(a^bb^cc^a)(a^2+1)^b(b^2+1)^c(c^2+1)^a}$$

$$\overset{\text{Weighted GM} \le \text{Weighted AM}}{\le} \frac{ab+bc+ca}{a+b+c} \cdot \frac{a^2b+b+b^2c+c+c^2a+a}{a+b+c}$$

$$\overset{\text{CBS}}{\le} \frac{\sum_{cyc} ab}{\sum_{cyc} a} \cdot \frac{\sqrt{(\sum_{cyc} a^2)(\sum_{cyc} a^2b^2)} + \sum_{cyc} a}{\sum_{cyc} a} \overset{?}{\le} \sqrt[a+b+c]{8} \overset{a+b+c=3}{=} 2$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)\left(\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} + \sum_{cyc} a\right) \overset{?}{\le} 2\left(\sum_{cyc} a\right)^2$$

$$\overset{\because a+b+c=3}{\Leftrightarrow} \left(\sum_{cyc} ab\right)\left(\begin{matrix}\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} \\ +\left(\sum_{cyc} a\right) \cdot \frac{1}{9}\left(\sum_{cyc} a\right)^2\end{matrix}\right) \overset{?}{\le} 2\left(\sum_{cyc} a\right)^2 \cdot \frac{1}{27}\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow 2\left(\sum_{cyc} a\right)^5 - 3\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^3 \overset{?}{\ge} 27\left(\sum_{cyc} ab\right) \cdot \sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}$$

$$\Leftrightarrow \boxed{\left(2\left(\sum_{cyc} a\right)^5 - 3\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^3\right)^2 \overset{?}{\ge} 729\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} ab\right)^2}$$

$$\overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow} (2s^5 - 3(4Rr+r^2)s^3)^2$$

$$\overset{?}{\ge} 729(s^2-8Rr-2r^2)\left(r^2((4R+r)^2-2s^2)\right)(4Rr+r^2)^2$$

$$\Leftrightarrow 4s^{10} - (48Rr+12r^2)s^8 + r^2(144R^2+72Rr+9r^2)s^6$$
$$+r^4(23328R^2+11664Rr+1458r^2)s^4$$
$$-r^4(186624R^4+373248R^3r+209952R^2r^2+46656Rr^3+3645r^4)s^2$$
$$+r^5\left(\begin{matrix}1492992R^5+1866240R^4r+933120R^3r^2+233280R^2r^3 \\ +29160Rr^4+1458r^5\end{matrix}\right) \boxed{\overset{?}{\underset{(*)}{\ge}}} 0 \text{ and}$$

$$\because 4(s^2-16Rr+5r^2)^5 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \ge 4(s^2-16Rr+5r^2)^5$$

$$\Leftrightarrow (272Rr-112r^2)s^8 - r^2(10096R^2-6472Rr+991r^2)s^6$$
$$+r^3(163840R^3-130272R^2r+59664Rr^2-3542r^3)s^4$$
$$-r^4(1497344R^4-1265152R^3r+977952R^2r^2-113344Rr^3+16145r^4)s^2$$
$$+r^5\left(\begin{matrix}5687296R^5-4687360R^4r+5029120R^3r^2-1046720R^2r^3 \\ +229160Rr^4-11042r^5\end{matrix}\right) \boxed{\overset{(**)}{\ge}} 0$$

$$\text{and} \because (272Rr-112r^2)(s^2-16Rr+5r^2)^4 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \ge (272Rr-112r^2)(s^2-16Rr+5r^2)^4$$

$$\Leftrightarrow (7312R^2-6136Rr+1249r^2)s^6$$

$$-r(253952R^3-302880R^2r+88656Rr^2-13258r^3)s^4$$
$$+r^2(2959104R^4-4747776R^3r+2047968R^2r^2-560256Rr^3+39855r^4)s^2$$
$$and \because (7312R^2-6136Rr+1249r^2)(s^2-16Rr+5r^2)^3 \overset{Gerretsen}{\geq} 0,$$
$$-r^3\begin{pmatrix}12138496R^5-24934912R^4r+14590720R^3r^2\\-5430080R^2r^3+836840Rr^4-58958r^5\end{pmatrix} \boxed{\overset{(***)}{\geq}} 0$$
$\therefore$ in order to prove $(***)$, it suffices to prove : LHS of $(***)$
$$\geq (7312R^2-6136Rr+1249r^2)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (97024R^3-101328R^2r+63336Rr^2-5477r^3)s^4$$
$$-r(2656512R^4-3474432R^3r+2404944R^2r^2-499464Rr^3+53820r^4)s^2$$
$$+r^3\begin{pmatrix}17811456R^5-28276224R^4r+22861824R^3r^2\\-7643280R^2r^3+1428960Rr^4-97167r^5\end{pmatrix} \boxed{\overset{(****)}{\geq}} 0 \ and$$
$$\because (97024R^3-101328R^2r+63336Rr^2-5477r^3)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0$$
$\therefore$ in order to prove $(****)$, it suffices to prove : LHS of $(****) \geq$
$$(97024R^3-101328R^2r+63336Rr^2-5477r^3)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow \boxed{\begin{matrix}(224128R^4-369152R^3r+317544R^2r^2-154580Rr^3+475r^4)s^2 \overset{(*****)}{\geq}\\ r(3513344R^5-6593792R^4r+5995136R^3r^2-3212896R^2r^3+515380Rr^4-19879r^5)\end{matrix}}$$
$$\text{Now, } (224128R^4-369152R^3r+317544R^2r^2-154580Rr^3+475r^4)s^2 \overset{Gerretsen}{\geq}$$
$$(224128R^4-369152R^3r+317544R^2r^2-154580Rr^3+475r^4)(16Rr-5r^2)$$
$$\overset{?}{\geq} r\begin{pmatrix}3513344R^5-6593792R^4r+5995136R^3r^2-\\3212896R^2r^3+515380Rr^4-19879r^5\end{pmatrix}$$
$$\Leftrightarrow \boxed{9088t^5-54160t^4+116416t^3-106013t^2+33140t+2188 \overset{?}{\geq} 0} \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)^2(184t^3+8904t^2(t-2)+8832t+547) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$
$$\Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore (a^3+a)^b(b^3+b)^c(c^3+c)^a \leq 8 \ \forall a,b,c>0 \mid a+b+c=3,$$
$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a^3+a=\frac{2a(a^2+1)}{2} \leq \frac{[2a+(a^2+1)]^2}{4.2}=\frac{(a+1)^4}{8} \quad \text{(and analogs).}$$

Then

$$(a^3+a)^b(b^3+b)^c(c^3+c)^a \leq \frac{(a+1)^{4b}(b+1)^{4c}(c+1)^{4a}}{8^{b+c+a}}$$

$$\overset{Weighted\ AM-GM}{\le} \frac{1}{8^3}\left(\frac{4b(a+1)+4c(b+1)+4a(c+1)}{4b+4c+4a}\right)^{4b+4c+4a} = \frac{1}{8^3}\left(\frac{ab+bc+ca}{3}+1\right)^{12}$$

$$\le \frac{1}{8^3}\left(\frac{(a+b+c)^2}{9}+1\right)^{12} = \frac{1}{8^3}.2^{12} = 8.$$

Equality holds iff $a=b=c=1$.

1452. If $a, b > 0$ and $a+b \ge \frac{1}{a^2}+\frac{1}{b^2}$, then prove that :

$$\frac{a^2}{a+1}+\frac{b^2}{b+1} \ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a+b \ge \frac{1}{a^2}+\frac{1}{b^2} \overset{Radon}{\ge} \frac{(1+1)^3}{(a+b)^2} \Rightarrow a+b \ge 2 \Rightarrow t = x+y \ge 4 \to (1)$$

$(x = a+1, y = b+1)$

$$\text{Now, } \frac{a^2}{a+1}+\frac{b^2}{b+1} \overset{?}{\ge} 1 \Leftrightarrow \frac{a^2-1+1}{a+1}+\frac{b^2-1+1}{b+1} \overset{?}{\ge} 1$$

$$\Leftrightarrow a-1+\frac{1}{a+1}+b-1+\frac{1}{b+1} \overset{?}{\ge} 1 \Leftrightarrow a+1+\frac{1}{a+1}+b+1+\frac{1}{b+1} \overset{?}{\ge} 5$$

$$\Leftrightarrow x+\frac{1}{x}+y+\frac{1}{y} \overset{?}{\ge} 5 \Leftrightarrow (x+y)\left(1+\frac{1}{xy}\right) \underset{(*)}{\overset{?}{\ge}} 5$$

$$\text{Now, } (x+y)\left(1+\frac{1}{xy}\right) \overset{A-G}{\ge} (x+y)\left(1+\frac{4}{(x+y)^2}\right) \overset{?}{\ge} 5 \Leftrightarrow t+\frac{4}{t} \overset{?}{\ge} 5$$

$$\Leftrightarrow t^2-5t+4 \overset{?}{\ge} 0 \Leftrightarrow (t-4)(t-1) \overset{?}{\ge} 0 \to \text{true} \because t \ge 4 \text{ via } (1) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a^2}{a+1}+\frac{b^2}{b+1} \ge 1\ \forall\, a, b > 0 \,\Big|\, a+b \ge \frac{1}{a^2}+\frac{1}{b^2}, '' = ''\ \text{iff } a = b = 1\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a+b \ge \frac{1}{a^2}+\frac{1}{b^2} \ge \frac{2}{ab} \ge \frac{8}{(a+b)^2} \Rightarrow a+b \ge 2.$$

By CBS inequality, we get

$$\frac{a^2}{a+1}+\frac{b^2}{b+1} \ge \frac{(a+b)^2}{a+b+2} \overset{a+b\ge 2}{\ge} \frac{2(a+b)}{a+b+2} \overset{a+b\ge 2}{\ge} \frac{(a+b)+2}{a+b+2} = 1.$$

Equality holds iff $a=b=1$.

1453.

If $a,b,c>0$ and $3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3$, then prove that :

$$\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}\le\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}$$

$$=\sum_{cyc}\frac{1}{\sqrt{(b-c)^2+3bc+c^2}}\le\sum_{cyc}\frac{1}{\sqrt{c(3b+c)}}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\sum_{cyc}\frac{1}{3b+c}}\overset{A-G}{\le}$$

$$\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\sum_{cyc}\frac{1}{2\sqrt{2b(b+c)}}}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\sum_{cyc}\frac{1}{2a}}\cdot\sqrt{\sum_{cyc}\frac{1}{b+c}}}\overset{\text{Reverse Bergstrom}}{\le}$$

$$\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\sum_{cyc}\frac{1}{2a}}\cdot\sqrt{\frac{1}{4}\sum_{cyc}\left(\frac{1}{b}+\frac{1}{c}\right)}}=\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sum_{cyc}\frac{1}{a}}}$$

$$=\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{4}\sum_{cyc}\frac{1}{a}}=\frac{1}{2}\sum_{cyc}\frac{1}{a}\overset{CBS}{\le}\frac{\sqrt{3}}{2}\sqrt{\sum_{cyc}\frac{1}{a^2}}$$

$$\therefore\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}\overset{(*)}{\le}\frac{\sqrt{3}}{2}\sqrt{\sum_{cyc}\frac{1}{a^2}}$$

$$\text{Now, }3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3\Rightarrow 3\sum_{cyc}\frac{1}{a^2}\le 2\sum_{cyc}\frac{1}{a^2}+3$$

$$\Rightarrow\sum_{cyc}\frac{1}{a^2}\overset{(**)}{\le}3\therefore(*),(**)\Rightarrow\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}$$

$$\le\frac{3}{2}\ \forall a,b,c>0\,\Big|\,3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3,$$

$$''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1454. If $a,b,c>0$ prove that:

$$\sqrt{\frac{1}{a}+\frac{1}{b}-\frac{2}{a+b+c}}+\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}+\sqrt{\frac{1}{c}+\frac{1}{a}-\frac{2}{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{ab^2+bc^2+ca^2}}{3\sqrt[6]{abc}}\ge 3$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}=\sum_{cyc}\sqrt{\frac{ab+ca+b^2+c^2}{bc(a+b+c)}}\geq\sum_{cyc}\sqrt{\frac{4\sqrt[4]{a^2b^3c^3}}{bc(a+b+c)}}$$

$$=\frac{2}{\sqrt{a+b+c}}\sum_{cyc}\sqrt[8]{\frac{a^2}{bc}}\geq\frac{2}{\sqrt{a+b+c}}.3\sqrt[3]{\sqrt[8]{\frac{a^2}{bc}}.\sqrt[8]{\frac{b^2}{ca}}.\sqrt[8]{\frac{c^2}{ab}}}=\frac{6}{\sqrt{a+b+c}}.$$

Also by AM − GM inequality, we have

$$ab^2+bc^2+ca^2=\sum_{cyc}\frac{2ca^2+ab^2}{3}\geq\sum_{cyc}\sqrt[3]{(ca^2)^2.ab^2}=\sqrt[3]{(abc)^2}.(a+b+c).$$

Therefore

$$\sum_{cyc}\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{ab^2+bc^2+ca^2}}{3\sqrt[6]{abc}}\geq\frac{6}{\sqrt{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{a+b+c}}{3}$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{\frac{6}{\sqrt{a+b+c}}.\left(\frac{\sqrt{6}.\sqrt[4]{a+b+c}}{2.3}\right)^2}=3$$

Equality holds iff $a=b=c=12$.

1455. If $a,b,c>0$, then prove that

$$\frac{a^4b^4(a^6+b^6)}{a^5+b^5}+\frac{b^4c^4(b^6+c^6)}{b^5+c^5}+\frac{c^4a^4(c^6+a^6)}{c^5+a^5}\geq\frac{1}{9}a^2b^2c^2(a+b+c)^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a^4b^4(a^6+b^6)}{a^5+b^5}\overset{Chebyshev}{\geq}\sum_{cyc}\frac{a^4b^4(a^5+b^5)(a+b)}{2(a^5+b^5)}=\sum_{cyc}\frac{a^4b^4(a+b)}{2}$$

$$=\frac{1}{2}\sum_{cyc}a^5(b^4+c^4)\overset{AM-GM}{\geq}\frac{1}{2}\sum_{cyc}a^5.2b^2c^2=$$

$$=\frac{1}{9}a^2b^2c^2.3^2\sum_{cyc}a^3\overset{Hölder}{\geq}\frac{1}{9}a^2b^2c^2(a+b+c)^3$$

Equality holds iff $a = b = c$.

1456. If $a, b, c > 0, abc = 1, n \in \mathbb{N}$ then:

$$\frac{a^{n+2}+b^{n+2}}{c^n(a^{n+1}+b^{n+1})}+\frac{b^{n+2}+c^{n+2}}{a^n(b^{n+1}+c^{n+1})}+\frac{c^{n+2}+a^{n+2}}{b^n(c^{n+1}+a^{n+1})}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^{n+2}+b^{n+2}}{c^n(a^{n+1}+b^{n+1})}\overset{LEHMER}{\geq}\sum_{cyc}\frac{a+b}{2c^n}\overset{AM-GM}{\geq}\sum_{cyc}\frac{\sqrt{ab}}{c^n}\geq$$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\sqrt{ab}}{c^n}\cdot\frac{\sqrt{bc}}{a^n}\cdot\frac{\sqrt{ca}}{b^n}}=3\sqrt[3]{\frac{abc}{(abc)^n}}=3\sqrt[3]{1}=3$$

Equality holds for $a = b = c = 1$.

1457. If $a, b, c > 0, abc = 1$ then:

$$\frac{b(b+c)^3+a(a+c)^3}{c(a+b)}+\frac{b(a+b)^3+c(a+c)^3}{a(b+c)}+\frac{a(a+b)^3+c(b+c)^3}{b(a+c)}\geq 24$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{b(b+c)^3+a(a+c)^3}{c(a+b)}+\frac{b(a+b)^3+c(a+c)^3}{a(b+c)}+\frac{a(a+b)^3+c(b+c)^3}{b(a+c)}=$$

$$=\frac{b(b+c)^3}{c(a+b)}+\frac{a(a+c)^3}{c(a+b)}+\frac{b(a+b)^3}{a(b+c)}+\frac{c(a+c)^3}{a(b+c)}+\frac{a(a+b)^3}{b(a+c)}+\frac{c(b+c)^3}{b(a+c)}\overset{AM-GM}{\geq}$$

$$6\cdot\sqrt[6]{\frac{b(b+c)^3}{c(a+b)}\cdot\frac{a(a+c)^3}{c(a+b)}\cdot\frac{b(a+b)^3}{a(b+c)}\cdot\frac{c(a+c)^3}{a(b+c)}\cdot\frac{a(a+b)^3}{b(a+c)}\cdot\frac{c(b+c)^3}{b(a+c)}}=$$

$$=6\cdot\sqrt[6]{\left(\prod_{cyc}(a+b)\right)^4}=6\cdot\sqrt[3]{\left(\prod_{cyc}(a+b)\right)^2}\overset{AM-GM}{\geq}6\cdot\sqrt[3]{\left(\prod_{cyc}(2\sqrt{ab})\right)^2}=$$

$$=6\cdot\sqrt[3]{64(abc)^2}=6\cdot\sqrt[3]{64}=6\cdot 4=24$$

Equality holds for: $a = b = c = 1$.

1458. (a) If $a, b, c > 0$, *then prove the inequality*:

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c| + |b-c|)^2 \text{ is false !}$$

(b) If $a, b, c > 0$, *then prove the inequality*:

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{8}\sum_{cyc}(|a-c| + |b-c|)^2 \text{ is true !}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c| + |b-c|)^2 - \sum_{cyc} a^2$$

$$= \sum_{cyc} ab - \sum_{cyc} a^2 + \frac{1}{2}\sum_{cyc}(a-c)^2 + \frac{1}{2}\sum_{cyc}(b-c)^2 + \sum_{cyc}|a-c||b-c|$$

$$= \sum_{cyc} ab - \sum_{cyc} a^2 + \frac{1}{2}.2\left(2\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + \sum_{cyc}|a-c||b-c|$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab + \sum_{cyc}|a-c||b-c| = \frac{1}{2}\sum_{cyc}(a-b)^2 + \sum_{cyc}|a-c||b-c| \geq 0$$

$$\therefore \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c| + |b-c|)^2 - \sum_{cyc} a^2 \geq 0$$

$$\Rightarrow \sum_{cyc} a^2 \leq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c| + |b-c|)^2$$

$$\Rightarrow \boxed{\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c| + |b-c|)^2 \text{ is false; reverse inequality is true}}$$

$$\text{Again, } \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\sum_{cyc}(|a-c| + |b-c|)^2$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\left(\sum_{cyc}(a-c)^2 + \sum_{cyc}(b-c)^2 + 2\sum_{cyc}|a-c||b-c|\right)$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\left(4\sum_{cyc} a^2 - 4\sum_{cyc} ab + 2\sum_{cyc}|a-c||b-c|\right)$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)-\frac{1}{4}\sum_{cyc}|a-c||b-c|=\frac{1}{4}\sum_{cyc}(a-b)^2-\frac{1}{4}\sum_{cyc}|a-c||b-c|$$

$$=\frac{1}{8}\left((|a-b|-|b-c|)^2+(|b-c|-|c-a|)^2+(|c-a|-|a-b|)^2\right)\geq 0$$

$$\therefore \sum_{cyc}a^2-\sum_{cyc}ab-\frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2\geq 0$$

$$\Rightarrow \boxed{\sum_{cyc}a^2\geq\sum_{cyc}ab+\frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2 \text{ is true}}, ''='' \text{ iff } a=b=c \text{ (QED)}$$

1459. If $a,b,c>0$ and $a^2+b^2+c^2=3$, then prove that :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\sum_{cyc}\frac{a^4}{a^2b}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b}\overset{\text{CBS}}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sqrt{\sum_{cyc}a^2b^2}\cdot\sqrt{\sum_{cyc}a^2}}$$

$$\overset{a^2+b^2+c^2=3}{=}\frac{\left(\sum_{cyc}a^2\right)^2}{\sqrt{3\sum_{cyc}a^2b^2}}\overset{\left(\sum_{cyc}a^2\right)^2\geq 3\sum_{cyc}a^2b^2}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2}=\sum_{cyc}a^2=3$$

$$\therefore \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3\ \forall\, a,b,c>0 \,\middle|\, a^2+b^2+c^2=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1460. If $x,y>0$ then:

$$\frac{1}{x}+\frac{1}{y}+\frac{2}{x+y}\geq\frac{3}{\sqrt{xy}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Eric Cismaru-Romania

Multiplying by $\sqrt{xy}$, the inequality is equivalent to

$$\frac{\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{y}}+\frac{2\sqrt{xy}}{x+y}\geq 0\Leftrightarrow\frac{y(x+y)+x(x+y)+2xy}{\sqrt{xy}(x+y)}\geq 3\Leftrightarrow$$

$$\Leftrightarrow (x+y)^2+2xy \geq 3\sqrt{xy}(x+y)$$

Making the substitution $s=x+y$ and $p=\sqrt{xy}$, with $s \geq 2p$ (from AM-GM), we have

$$s^2+2p^2-3ps = s^2-ps+2p^2-2ps = s(s-p)+2p(p-s) = (s-p)(s-2p) \geq 0$$

which is always true.

Equality holds if and only if $x=y$.

1461. Let $a,b,c \geq 0$ and $a^2+b^2+c^2 \neq 0$. Prove that :

$$\sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} \geq \frac{a+b+c}{2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 : Exactly one variable $=0$ and WLOG we may assume $a=0$

$$\text{and then}: \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} = \frac{b^3+c^3}{b^2+c^2} \overset{\text{Chebyshev}}{\geq} \frac{\frac{1}{2}(b+c)(b^2+c^2)}{b^2+c^2} = \frac{b+c}{2}$$

$$\therefore \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} \geq \frac{a+b+c}{2}$$

Case 2 : Exactly two variables $=0$ and WLOG we may assume $b=c=0$

$$\text{and then}: \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} = \frac{a^3}{a^2} = a > \frac{a}{2}$$

$$\Rightarrow \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} > \frac{a+b+c}{2}$$

Case 3 : $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z$

$\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0$

$\Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle

with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a =$

$$\sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b=s-y, c=s-z \therefore abc = r^2 s \to (2)$$

$$\text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr+r^2 \to (3),$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2-8Rr-2r^2 \to (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} \left(4Rr + r^2\right)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5),$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a + b)(b + c)(c + a) \overset{\text{via (1)}}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} a^3 = s\left(s^2 - 12Rr\right) \to (6) \text{ and } \sum_{cyc} a^4 = \left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2 \overset{\text{via (4) and (5)}}{=}$$

$$\left(s^2 - 8Rr - 2r^2\right)^2 - 2r^2\left((4R + r)^2 - 2s^2\right)$$

$$\Rightarrow \sum_{cyc} a^4 = \left(s^2 - 8Rr - 2r^2\right)^2 - 2r^2\left((4R + r)^2 - 2s^2\right) \to (7)$$

$$\text{Now, } \sum_{cyc} \frac{a(a^2 + 3bc)}{(b + c)^2 + a^2 + 3bc} = \sum_{cyc} \frac{a^3}{\sum_{cyc} a^2 + 5bc} + 3abc\sum_{cyc} \frac{1}{\sum_{cyc} a^2 + 5bc} \to (\bullet)$$

$$\text{Firstly, } \sum_{cyc}\left(a^3\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= \sum_{cyc}\left(a^3\left(\left(\sum_{cyc} a^2\right)^2 + 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab - bc\right) + 25abc.a\right)\right)$$

$$= \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^2\right)^2 + 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^3\right) - 5abc\left(\sum_{cyc} a^2\right)^2$$

$$+25abc\sum_{cyc} a^4 \overset{\text{via (2),(3),(4),(6) and (7)}}{=}$$

$$s\left(s^2 - 12Rr\right)\left(s^2 - 8Rr - 2r^2\right)^2 + 5\left(s^2 - 8Rr - 2r^2\right)\left(4Rr + r^2\right).s\left(s^2 - 12Rr\right)$$

$$-5r^2s\left(s^2 - 8Rr - 2r^2\right)^2 + 25r^2s\left(\left(s^2 - 8Rr - 2r^2\right)^2 - 2r^2\left((4R + r)^2 - 2s^2\right)\right)$$

$$\therefore \sum_{cyc}\left(a^3\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= s\begin{pmatrix} s^6 - \left(8Rr - 21r^2\right)s^4 - \left(144R^2 + 380Rr - 14r^2\right)r^2s^2 \\ +r^3\left(1152R^3 + 1056R^2r + 312Rr^2 + 30r^3\right) \end{pmatrix} \to (i)$$

$$\text{Secondly, } \sum_{cyc}\left(\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= 3\left(\sum_{cyc} a^2\right)^2 + 10\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + 25abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via }(1),(2),(3)\text{ and }(4)}{=} 3(s^2 - 8Rr - 2r^2)^2 + 10(4Rr + r^2)(s^2 - 8Rr - 2r^2) + 25r^2s^2$$

$$\therefore \sum_{cyc}\left(\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right) = 3s^4 - (8Rr - 23r^2)s^2 - 8r^2(4R + r)^2 \to \text{(ii)}$$

$$\text{Thirdly, }\left(\sum_{cyc} a^2 + 5bc\right)\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 5\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} ab\right) + 25abc\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right) + 125(abc)^2$$

$$\overset{\text{via }(1),(2),(3)\text{ and }(4)}{=} (s^2 - 8Rr - 2r^2)^3 + 5(4Rr + r^2)(s^2 - 8Rr - 2r^2)^2$$
$$+25r^2s^2(s^2 - 8Rr - 2r^2) + 125r^4s^2$$

$$\therefore \left(\sum_{cyc} a^2 + 5bc\right)\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)$$
$$= s^6 - (4Rr - 24r^2)s^4 - (128R^2 + 264Rr - 67r^2)r^2s^2 + 12r^3(4R + r)^3 \to \text{(iii)}$$

$$\text{Now, }(\bullet), \text{(i), (ii), (iii)} \Rightarrow \sum_{cyc} \frac{a(a^2 + 3bc)}{(b + c)^2 + a^2 + 3bc} \ge \frac{a + b + c}{2}$$

$$\Leftrightarrow s.\frac{\begin{array}{c} s^6 - (8Rr - 21r^2)s^4 - (144R^2 + 380Rr - 14r^2)r^2s^2 \\ +r^3(1152R^3 + 1056R^2r + 312Rr^2 + 30r^3) \end{array}}{s^6 - (4Rr - 24r^2)s^4 - (128R^2 + 264Rr - 67r^2)r^2s^2 + 12r^3(4R + r)^3}$$
$$+3r^2s.\frac{3s^4 - (8Rr - 23r^2)s^2 - 8r^2(4R + r)^2}{s^6 - (4Rr - 24r^2)s^4 - (128R^2 + 264Rr - 67r^2)r^2s^2 + 12r^3(4R + r)^3} \ge \frac{s}{2}$$

$$\Leftrightarrow \boxed{s^6 - (12Rr - 36r^2)s^4 - (160R^2 + 544Rr - 99r^2)r^2s^2 + 96Rr^3(4R + r)^2 \overset{(*)}{\ge} 0}$$

$$and \because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove }(*), \text{it suffices to prove :}$$

$$\text{LHS of }(*) \ge (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow \boxed{\begin{array}{c} (36Rr + 21r^2)s^4 - (928R^2 + 64Rr - 24r^2)r^2s^2 \\ +r^3(5632R^3 - 3072R^2r + 1296Rr^2 - 125r^3) \overset{(**)}{\ge} 0 \end{array}} \text{ and}$$

$$\because (36Rr + 21r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove }(**),$$
$$\text{it suffices to prove : LHS of }(**) \ge (36Rr + 21r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow \boxed{(112R^2 + 124Rr - 93r^2)s^2 \overset{(***)}{\ge} r(1792R^3 + 1344R^2r - 1878Rr^2 + 325r^3)}$$

$$\text{Again, }(112R^2 + 124Rr - 93r^2)s^2 \overset{\text{Gerretsen}}{\ge}$$
$$(112R^2 + 124Rr - 93r^2)(16Rr - 5r^2) \overset{?}{\ge}$$

$$r\left(1792R^3+1344R^2r-1878Rr^2+325r^3\right)\Leftrightarrow 10r\left(8R^2-23Rr+14r^2\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 10r(R-2r)(8R-7r)\overset{?}{\geq}0\to \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \sum_{\text{cyc}}\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\geq\frac{a+b+c}{2}\therefore \text{combining all cases,}$$

$$\sum_{\text{cyc}}\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\geq\frac{a+b+c}{2}\ \forall\, a,b,c\geq 0 \text{ and } a^2+b^2+c^2\neq 0,$$

$$''='' \text{ iff } (a=0,b=c>0) \text{ or } (b=0,c=a>0) \text{ or } (c=0,a=b>0)$$
$$\text{or } (a=b=c>0)\ (\text{QED})$$

1462. If $a,b,c>0$, *then* :

$$\left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2+2\sum_{\text{cyc}}a\left(\frac{1}{b}+\frac{1}{c}\right)\geq 15$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius
$$=s,R,r\ (say)\text{ yielding } 2\sum_{\text{cyc}}a=\sum_{\text{cyc}}x=2s\Rightarrow\sum_{\text{cyc}}a=s\to(1)$$

$$\Rightarrow a=s-x, b=s-y, c=s-z\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow$$
$$\sum_{\text{cyc}}ab=\sum_{\text{cyc}}(s-x)(s-y)\Rightarrow\sum_{\text{cyc}}ab=4Rr+r^2\to(3)$$

$$\therefore\left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2+2\sum_{\text{cyc}}a\left(\frac{1}{b}+\frac{1}{c}\right)\geq 15$$

$$\overset{\text{via }(1),(2)\text{ and }(3)}{\Leftrightarrow}\frac{243r^4s^2}{s^2(4Rr+r^2)^2}+2.\sum_{\text{cyc}}\frac{a^2(b+c)}{abc}\geq 15$$

$$\Leftrightarrow\frac{243r^2}{(4R+r)^2}+2.\frac{\sum_{\text{cyc}}\left(ab\left(\sum_{\text{cyc}}a-c\right)\right)}{abc}\geq 15$$

$$\Leftrightarrow\frac{243r^2}{(4R+r)^2}+2.\frac{\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}ab\right)-3abc}{abc}\geq 15$$

$$\overset{\text{via }(1),(2)\text{ and }(3)}{\Leftrightarrow}\frac{243r^2}{(4R+r)^2}+2.\frac{s\left(4Rr+r^2\right)-3r^2s}{r^2s}\geq 15$$

$$\Leftrightarrow 2(4R-2r)(4R+r)^2+243r^3\geq 15r(4R+r)^2\Leftrightarrow 8t^3-15t^2-9t+14\geq 0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(8\left(t^2-4\right)+t+25\right)\geq 0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2+2\sum_{cyc}a\left(\frac{1}{b}+\frac{1}{c}\right)\geq 15\ \forall\, a,b,c>0,$$

$$''=''\ \text{ iff } a=b=c\ (\text{QED})$$

1463. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{b(b+c)^{3n}+a(a+c)^{3n}}{c(a+b)}+\frac{b(a+b)^{3n}+c(a+c)^{3n}}{a(b+c)}+\frac{a(a+b)^{3n}+c(b+c)^{3n}}{b(a+c)}\geq 3.8^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{b(b+c)^{3n}+a(a+c)^{3n}}{c(a+b)}+\frac{b(a+b)^{3n}+c(a+c)^{3n}}{a(b+c)}$$

$$+\frac{a(a+b)^{3n}+c(b+c)^{3n}}{b(a+c)}=\sum_{cyc}\left((b+c)^{3n}\left(\frac{b}{c(a+b)}+\frac{c}{b(c+a)}\right)\right)$$

$$=\sum_{cyc}\left(\frac{(b+c)^{3n}}{(a+b)(c+a)}\cdot\frac{b^2c+b^2a+c^2a+c^2b}{bc}\right)$$

$$\geq\sum_{cyc}\left(\frac{(b+c)^{3n}}{(a+b)(c+a)}\cdot\frac{bc(b+c)+\frac{a(b+c)^2}{2}}{bc}\right)$$

$$=\sum_{cyc}\left(\frac{(b+c)^{3n}}{2}\cdot\frac{(2bc+ab+ac)(b+c)}{(a+b)(c+a)bc}\right)$$

$$\overset{A-G}{\geq}3.\sqrt[3]{\frac{(a+b)^{3n}.(b+c)^{3n}.(c+a)^{3n}}{8}\cdot\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{a^2b^2c^2(a+b)(b+c)(c+a)}}$$

$$\overset{abc=1}{=}3\left(\frac{(a+b)(b+c)}{(c+a)}\right)^n\cdot\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}$$

$$\overset{\text{Cesaro and }\because n\geq 1}{\geq}3(8abc)^n.\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}$$

$$\overset{abc=1}{=}3.8^n.\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}\overset{?}{\geq}3.8^n$$

$$\Leftrightarrow(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)\overset{?}{\geq}8abc(a+b)(b+c)(c+a)$$

$$\Leftrightarrow 2\sum_{cyc}a^3b^3\geq abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\Leftrightarrow 2\sum_{cyc} a^3b^3 + 3a^2b^2c^2 \ge abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) + 3a^2b^2c^2 \to \text{true}$$

$$\because \sum_{cyc} a^3b^3 + 3a^2b^2c^2 \overset{\text{Schur}}{\ge} abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \text{ and } \sum_{cyc} a^3b^3 \overset{A-G}{\ge} 3a^2b^2c^2$$

$$\therefore \frac{b(b+c)^{3n} + a(a+c)^{3n}}{c(a+b)} + \frac{b(a+b)^{3n} + c(a+c)^{3n}}{a(b+c)} + \frac{a(a+b)^{3n} + c(b+c)^{3n}}{b(a+c)}$$
$$\ge 3.8^n\ \forall a,b,c > 0, ''='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1464. If $a,b,c > 0$ such that $abc = 1$, then prove that :

$$\frac{a^5b^5(a^2+b^2)}{c^5(a^6+b^6)} + \frac{b^5c^5(b^2+c^2)}{a^5(b^6+c^6)} + \frac{c^5a^5(c^2+a^2)}{b^5(c^6+a^6)} \ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove that $\forall\ \Delta ABC$,

$$\boxed{(s^2 - 8Rr - 2r^2)^6 \ge 27r^5s^4(4R - 5r)^3 \to (\bullet)} \text{ and}$$

$$\because 10240R^3 - 26880R^2r + 23520Rr^2 - 6860r^3$$
$$= (R - 2r)(10240R^2 - 6400Rr + 10720r^2) + 14580r^3 \overset{\text{Euler}}{\ge} 14580r^3$$
$$> 0 \text{ and } 61440R^4 - 216768R^3r + 288720R^2r^2 - 172740Rr^3 + 39390r^4$$
$$= (R - 2r)(61440R^3 - 93888R^2r + 100944Rr^2 + 29148r^3) + 97686r^4$$
$$\overset{\text{Euler}}{\ge} 97686r^4 > 0 \therefore \text{via Gerretsen,}$$

$$\boxed{P} = (s^2 - 16Rr + 5r^2)^6 + (48R - 42r^2)(s^2 - 16Rr + 5r^2)^5$$
$$+r^2(960R^2 - 1680Rr + 735r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+r^3(10240R^3 - 26880R^2r + 23520Rr^2 - 6860r^3)(s^2 - 16Rr + 5r^2)^3$$
$$+r^4\begin{pmatrix} 61440R^4 - 216768R^3r + 288720R^2r^2 \\ -172740Rr^3 + 39390r^4 \end{pmatrix}(s^2 - 16Rr + 5r^2)^2 \ge 0$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove :

$$(s^2 - 8Rr - 2r^2)^6 - 27r^5s^4(4R - 5r)^3 \ge P$$

$$\Leftrightarrow \begin{pmatrix} 49152R^5 - 228864R^4r + 432480R^3r^2 - 410280R^2r^3 \\ +191310Rr^4 - 33648r^5 \end{pmatrix}s^2$$
$$\boxed{\overset{(*)}{\ge}}\ r\begin{pmatrix} 720896R^6 - 3452928R^5r + 6827520R^4r^2 - 7060400R^3r^3 \\ +3955620R^2r^4 - 1107609Rr^5 + 117734r^6 \end{pmatrix}$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} \begin{pmatrix} 49152R^5 - 228864R^4r + 432480R^3r^2 \\ -410280R^2r^3 + 191310Rr^4 - 33648r^5 \end{pmatrix}(16Rr - 5r^2)$$
$$\overset{?}{\ge} r\begin{pmatrix} 720896R^6 - 3452928R^5r + 6827520R^4r^2 - 7060400R^3r^3 \\ +3955620R^2r^4 - 1107609Rr^5 + 117734r^6 \end{pmatrix}$$
$$\Leftrightarrow 65536t^6 - 454656t^5 + 1236480t^4 - 1666480t^3 + 1156740t^2$$

$$-387309t + 50506 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{(t-2)\left(\begin{matrix}(t-2)\left((t-2)(65536t^3 - 61440t^2 + 81408t + 83536) + 189540\right) \\ +19683\end{matrix}\right) \overset{?}{\geq} 0}$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \Rightarrow (\bullet) \text{ is true}$$

$$\text{Now, } \frac{a^5b^5(a^2+b^2)}{c^5(a^6+b^6)} + \frac{b^5c^5(b^2+c^2)}{a^5(b^6+c^6)} + \frac{c^5a^5(c^2+a^2)}{b^5(c^6+a^6)} \overset{abc=1}{=}$$

$$\sum_{cyc} \frac{b^8c^8(b^2+c^2)}{a^2(b^2+c^2)(b^4-b^2c^2+c^4)} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} b^4c^4\right)^2}{\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 6a^2b^2c^2}$$

$$= \frac{\left(\sum_{cyc} x^2y^2\right)^2}{\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 6xyz} \left(x = a^2, y = b^2, z = c^2\right) \overset{?}{\geq} 3 \overset{xyz=1}{=} 3(xyz)^{\frac{5}{3}}$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc} x^2y^2\right)^6 \underbrace{\overset{?}{\geq}}_{(**)} 27(xyz)^5.\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 6xyz\right)^3}$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \rightarrow (i)$$

$$\Rightarrow x = s - X, y = s - Y, z = s - Z \text{ and such substitutions} \Rightarrow$$

$$\sum_{cyc} xy = \sum_{cyc} (s-X)(s-Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \rightarrow (ii) \text{ and}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (i) and (ii)}}{=}$$

$$(4Rr + r^2)^2 - 2\left(\prod_{cyc} (s-X)\right).s$$

$$= (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R+r)^2 - 2s^2\right) \rightarrow (iii)$$

$$\therefore \text{via (i), (ii) and (iii), } (**) \Leftrightarrow$$

$$r^{12}\left((4R+r)^2 - 2s^2\right)^6 \geq 27r^{10}s^5\left(s(4Rr + r^2) - 6r^2s\right)^3$$

$$\Leftrightarrow \left((4R+r)^2 - 2s^2\right)^6 \geq 27rs^2.\left(4R + r - \frac{6rs^2}{s^2}\right)^3.s^6$$

$$\Leftrightarrow \left(\sum_{cyc} r_a^2\right)^6 \geq 27r_ar_br_c.\left(\sum_{cyc} r_a - \frac{6r_ar_br_c}{\sum_{cyc} r_ar_b}\right)^3.\left(\sum_{cyc} r_ar_b\right)^3$$

$$\Leftrightarrow\left(\sum_{cyc}r_a^2\right)^6\ge 27r_ar_br_c\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-6r_ar_br_c\right)^3$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc}x'^2\right)^6\overset{(***)}{\ge}27x'y'z'\left(\left(\sum_{cyc}x'\right)\left(\sum_{cyc}x'y'\right)-6x'y'z'\right)^3}\left(\begin{matrix}x'=r_a,\\ y'=r_b,z'=r_c\end{matrix}\right)$$

Assigning $y'+z'=X', z'+x'=Y', x'+y'=Z'\Rightarrow X'+Y'-Z'=2z'$ $>0, Y'+Z'-X'=2x'>0$ and $Z'+X'-Y'=2y'>0\Rightarrow X'+Y'>Z'$, $Y'+Z'>X', Z'+X'>Y'\Rightarrow X',Y',Z'$ *form sides of* a triangle with semiperimeter, circumradius and inradius $=s',R',r'$ (say) yielding

$$2\sum_{cyc}x'=\sum_{cyc}X'=2s'\Rightarrow\sum_{cyc}x'=s'\to(1)$$

$\Rightarrow x'=s'-X', y'=s'-Y', z'=s'-Z'$ and such substitutions

$$\Rightarrow\sum_{cyc}x'y'=\sum_{cyc}(s'-X')(s'-Y')\Rightarrow\sum_{cyc}x'y'=4R'r'+r'^2\to(2)\text{ and}$$

$$\sum_{cyc}x'^2=\left(\sum_{cyc}x'\right)^2-2\sum_{cyc}x'y'\overset{\text{via (1) and (2)}}{=}s'^2-2\left(4R'r'+r'^2\right)$$

$$\Rightarrow\sum_{cyc}x'^2=s'^2-8R'r'-2r'^2\to(3)\therefore\text{ via (1), (2) and (3), }(***)$$

$$\Leftrightarrow\left(s'^2-8R'r'-2r'^2\right)^6\ge 27r'^2s'\left(s'\left(4R'r'+r'^2\right)-6r'^2s'\right)^3$$

$$\Leftrightarrow\boxed{\left(s'^2-8R'r'-2r'^2\right)^6\ge 27r'^5s'^4(4R'-5r')^3}\to\text{true via }(\bullet)$$

$$\Rightarrow(***)\Rightarrow(**)\text{ is true }\therefore\frac{a^5b^5(a^2+b^2)}{c^5(a^6+b^6)}+\frac{b^5c^5(b^2+c^2)}{a^5(b^6+c^6)}+\frac{c^5a^5(c^2+a^2)}{b^5(c^6+a^6)}\ge 3$$

$\forall\, a,b,c>0\,|\,abc=1,''=''$ iff $a=b=c=1$ (QED)

1465. If $a,b,c>0$, *then prove that*:

$$\sum_{cyc}\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\ge 1$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\ge 1\Leftrightarrow\sum_{cyc}\frac{2a^2+c(b+c-2c)}{(b+c)(a+b+c)}\ge 1$$

$$\Leftrightarrow 2\sum_{cyc}\frac{a^2}{b+c}+\sum_{cyc}c-2\sum_{cyc}\frac{c^2}{b+c}\ge\sum_{cyc}a\Leftrightarrow\sum_{cyc}\frac{(a+c)(a-c)}{b+c}\ge 0\to(*)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a$

$$> 0 \text{ and } z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z \text{ form sides}$$
$$\text{of a triangle with semiperimeter, circumradius and inradius} = s, R, r \text{ (say)}$$
$$\text{yielding } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s - x, b = s - y,$$
$$c = s - z \text{ and via such substitutions, } (*) \Leftrightarrow \sum_{cyc} \frac{y\big((s-x)-(s-z)\big)}{x} \geq 0$$
$$\Leftrightarrow \sum_{cyc} \frac{y(z-x)}{x} \geq 0 \Leftrightarrow \sum_{cyc} \frac{yz}{x} \geq \sum_{cyc} y \Leftrightarrow \sum_{cyc} x^2y^2 \geq xyz\sum_{cyc} x \to \text{true} \Rightarrow (*) \text{ is true}$$
$$\therefore \sum_{cyc} \frac{2a^2 + c(b-c)}{(b+c)(a+b+c)} \geq 1 \ \forall a,b,c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

1466. If $a_k > 0$ $(k = 1, 2, \dots, n)$, then prove that:

$$16\sum_{cyc} \frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)} + \sum_{cyc} \frac{a_1^2+a_2^2}{a_1+a_2} \leq 3\sum_{k=1}^{n} a_k$$

Proposed by Mihaly Bencze, Neculai Stanciu –Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{16xy(x+y)}{(3x+y)(x+3y)} + \frac{x^2+y^2}{x+y} \overset{?}{\leq} \frac{3(x+y)}{2}$$
$$\Leftrightarrow 3(3x+y)(x+3y)(x+y)^2 \overset{?}{\geq} 2\left(16xy(x+y)^2 + (3x+y)(x+3y)(x^2+y^2)\right)$$
$$\Leftrightarrow 3x^4 + 3y^4 + 2x^2y^2 - 4xy(x^2+y^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow (x^4+y^4+2x^2y^2) + 2(x^4+y^4) - 4xy(x^2+y^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow (x^2+y^2)^2 + (x^2+y^2)^2 + (x^2-y^2)^2 - 4xy(x^2+y^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow 2(x^2+y^2)(x^2+y^2-2xy) + (x^2-y^2)^2 \overset{?}{\geq} 0$$
$$\Leftrightarrow 2(x^2+y^2)(x-y)^2 + (x^2-y^2)^2 \overset{?}{\geq} 0 \to \text{true}$$
$$\therefore \frac{16xy(x+y)}{(3x+y)(x+3y)} + \frac{x^2+y^2}{x+y} \leq \frac{3(x+y)}{2}$$
$$\therefore \frac{16\, a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)} + \frac{a_1^2+a_2^2}{a_1+a_2} \leq \frac{3(a_1+a_2)}{2} \text{ and analogs}$$
$$\Rightarrow 16\sum_{cyc} \frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)} + \sum_{cyc} \frac{a_1^2+a_2^2}{a_1+a_2} \leq 3\sum_{k=1}^{n} a_k$$
$$\forall a_k > 0 \ (k = 1, 2, \dots, n), '' ='' \text{ iff } a_1 = a_2 = \dots = a_n \text{ (QED)}$$

1467. If $a_k > 0$ $(k = 1, 2, \dots, n)$, then prove that :

$$16\sum_{cyc}\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum_{cyc}\frac{a_1^2+a_2^2}{a_1+a_2}\le 3\sum_{k=1}^{n}a_k$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{16xy(x+y)}{(3x+y)(x+3y)}+\frac{x^2+y^2}{x+y}\overset{?}{\le}\frac{3(x+y)}{2}$$

$$\Leftrightarrow 3(3x+y)(x+3y)(x+y)^2\overset{?}{\ge}2\left(16xy(x+y)^2+(3x+y)(x+3y)(x^2+y^2)\right)$$

$$\Leftrightarrow 3x^4+3y^4+2x^2y^2-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow (x^4+y^4+2x^2y^2)+2(x^4+y^4)-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow (x^2+y^2)^2+(x^2+y^2)^2+(x^2-y^2)^2-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 2(x^2+y^2)(x^2+y^2-2xy)+(x^2-y^2)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 2(x^2+y^2)(x-y)^2+(x^2-y^2)^2\overset{?}{\ge}0\to \text{true}$$

$$\therefore \frac{16xy(x+y)}{(3x+y)(x+3y)}+\frac{x^2+y^2}{x+y}\le\frac{3(x+y)}{2}$$

$$\therefore \frac{16\,a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\frac{a_1^2+a_2^2}{a_1+a_2}\le\frac{3(a_1+a_2)}{2}\text{ and analogs}$$

$$\Rightarrow 16\sum_{cyc}\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum_{cyc}\frac{a_1^2+a_2^2}{a_1+a_2}\le 3\sum_{k=1}^{n}a_k$$

$$\forall a_k>0\ (k=1,2,\dots,n),''='' \text{ iff } a_1=a_2=\dots=a_n\ (QED)$$

1468. If $a, b, c > 0$, *then prove that* :

$$\sum_{cyc}\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\ge 1$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\ge 1\Leftrightarrow\sum_{cyc}\frac{2a^2+c(b+c-2c)}{(b+c)(a+b+c)}\ge 1$$

$$\Leftrightarrow 2\sum_{cyc}\frac{a^2}{b+c}+\sum_{cyc}c-2\sum_{cyc}\frac{c^2}{b+c}\geq\sum_{cyc}a\Leftrightarrow\sum_{cyc}\frac{(a+c)(a-c)}{b+c}\geq 0\to(*)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\Rightarrow a=s-x, b=s-y,$

$c=s-z$ and via such substitutions, $(*)\Leftrightarrow\sum_{cyc}\frac{y\big((s-x)-(s-z)\big)}{x}\geq 0$

$$\Leftrightarrow\sum_{cyc}\frac{y(z-x)}{x}\geq 0\Leftrightarrow\sum_{cyc}\frac{yz}{x}\geq\sum_{cyc}y\Leftrightarrow\sum_{cyc}x^2y^2\geq xyz\sum_{cyc}x\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\geq 1\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1469. If $a,b,c>0$ and $abc=1$, then prove that $\forall\, n\in\mathbb{N}$, we have:

$$\frac{b(b^2+bc+c^2)^{3n}+a(a^2+ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}+\frac{a(a^2+ab+b^2)^{3n}+c(b^2+bc+c^2)^{3n}}{b(a+c)}\geq 3^{3n+1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}\overset{A-G}{\geq}\sum_{cyc}\frac{b(3ab)^{3n}+c(3ac)^{3n}}{a(b+c)}$$

$$=\sum_{cyc}\left(\frac{3^{3n}.a^{3n-1}}{b+c}.(b.b^{3n}+c.c^{3n})\right)\overset{Chebyshev}{\geq}\sum_{cyc}\left(\frac{3^{3n}.a^{3n-1}}{2(b+c)}.(b+c)(b^{3n}+c^{3n})\right)$$

$$=\frac{3^{3n}}{2}.\sum_{cyc}a^{3n-1}.(b^{3n}+c^{3n})\overset{A-G}{\geq}\frac{3.3^{3n}}{2}.\sqrt[3]{(abc)^{3n-1}.\prod_{cyc}(b^{3n}+c^{3n})}$$

$$\overset{Cesaro}{\geq}\frac{3.3^{3n}}{2}.\sqrt[3]{(abc)^{3n-1}.8(abc)^{3n}}\overset{abc=1}{=}\frac{3.3^{3n}.2}{2}=3^{3n+1}$$

$$\therefore\frac{b(b^2+bc+c^2)^{3n}+a(a^2+ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}+\frac{a(a^2+ab+b^2)^{3n}+c(b^2+bc+c^2)^{3n}}{b(a+c)}\geq 3^{3n+1}\ \forall\, a,b,c>0\,|\,abc=1$$

and $\forall\, n\in\mathbb{N},''='' \text{ iff } a=b=c=1$ (QED)

1470. If $a, b, c > 0$ and $abc = 1$, then prove that $\forall\, n \in \mathbb{N}$, we have :

$$\frac{b(b^2-bc+c^2)^{3n}+a(a^2-ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2-ab+b^2)^{3n}+c(a^2-ac+c^2)^{3n}}{a(b+c)}$$
$$++\frac{a(a^2-ab+b^2)^{3n}+c(b^2-bc+c^2)^{3n}}{b(a+c)} \geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b(a^2-ab+b^2)^{3n}+c(a^2-ac+c^2)^{3n}}{a(b+c)} \overset{A-G}{\geq} \sum_{cyc}\frac{b(ab)^{3n}+c(ac)^{3n}}{a(b+c)}$$
$$=\sum_{cyc}\left(\frac{a^{3n-1}}{b+c}.\left(b.b^{3n}+c.c^{3n}\right)\right) \overset{Chebyshev}{\geq} \sum_{cyc}\left(\frac{a^{3n-1}}{2(b+c)}.(b+c)\left(b^{3n}+c^{3n}\right)\right)$$
$$=\frac{1}{2}.\sum_{cyc}a^{3n-1}.\left(b^{3n}+c^{3n}\right) \overset{A-G}{\geq} \frac{3}{2}.\sqrt[3]{(abc)^{3n-1}.\prod_{cyc}(b^{3n}+c^{3n})}$$
$$\overset{Cesaro}{\geq} \frac{3}{2}.\sqrt[3]{(abc)^{3n-1}.8(abc)^{3n}} \overset{abc=1}{=} \frac{3.2}{2} = 3$$
$$\therefore \frac{b(b^2-bc+c^2)^{3n}+a(a^2-ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2-ab+b^2)^{3n}+c(a^2-ac+c^2)^{3n}}{a(b+c)}$$
$$+\frac{a(a^2-ab+b^2)^{3n}+c(b^2-bc+c^2)^{3n}}{b(a+c)} \geq 3\ \forall\, a,b,c > 0 \,|\, abc = 1 \text{ and } \forall\, n \in \mathbb{N},$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1471. If a,b,c > 0 ,then prove that

$$\sum\frac{a}{b} - \frac{\sum a^2}{\sum ab} \geq 2$$

Proposed by Neculai Stanciu-Romania

Solution by Cosghun Memmedov-Azerbaijan

$$\sum\frac{a}{b} = \sum\frac{a^2}{ab} \overset{Bergstrom}{\geq} \frac{(\sum a)^2}{\sum ab} = \frac{\sum a^2+2\sum ab}{\sum ab} = 2+\frac{\sum a^2}{\sum ab} \Rightarrow \sum\frac{a}{b} - \frac{\sum a^2}{\sum ab} \geq 2$$

1472. If $x,y,z>0$ such that : $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 3$, then :

$$\sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5}\ge \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 3 \overset{\text{via A-G}}{\Rightarrow} 3\ge 3.\sqrt[3]{\frac{1}{x}.\frac{1}{y}.\frac{1}{z}}\Rightarrow xyz\ge 1\to (1)$$

$$\text{Now,}\left(\sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5}\right)^2=\sum_{\text{cyc}}\left(\frac{yz}{x}\right)^5+2\sum_{\text{cyc}}\sqrt{\left(\frac{xy}{z}\right)^5}.\sqrt{\left(\frac{zx}{y}\right)^5}$$

$$\overset{\because yz\ge\frac{1}{x}\text{ and analogs ... via (1)}}{\ge}\sum_{\text{cyc}}\frac{1}{x^{10}}+2\sum_{\text{cyc}}x^5=\sum_{\text{cyc}}\left(\frac{1}{x^{10}}+x^5+x^5\right)\overset{\text{A-G}}{\ge}$$

$$\sum_{\text{cyc}}3.\sqrt[3]{\frac{1}{x^{10}}.x^5.x^5}=9\ge\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\left(\because 3\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\therefore\sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5}\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z},''=''\text{ iff } x=y=z=1\text{ (QED)}$$

1473.

If $a,b,c>0$ such that : $a+b+c=3$, then :

$$\frac{9}{2}+\sqrt{\left(\frac{ab}{c}\right)^5}+\sqrt{\left(\frac{bc}{a}\right)^5}+\sqrt{\left(\frac{ca}{b}\right)^5}\ge\frac{45}{2}.\frac{abc}{ab+bc+ca}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}ab\overset{\text{A-G}}{\ge}3.\sqrt[3]{a^2b^2c^2}\overset{?}{\ge}3abc\Leftrightarrow abc\overset{?}{\le}1\to\text{true}\because 3=\sum_{\text{cyc}}a\overset{\text{A-G}}{\ge}3.\sqrt[3]{abc}$$

$$\Rightarrow abc\le 1\Rightarrow\frac{abc}{ab+bc+ca}\le\frac{1}{3}\Rightarrow\frac{45}{2}.\frac{abc}{ab+bc+ca}-\frac{9}{2}\le 3\to(1)$$

$$\text{Let }\frac{bc}{a}=x,\frac{ca}{b}=y,\frac{ab}{c}=z\Rightarrow abc=xyz\Rightarrow a^2=yz,b^2=zx,c^2=xy$$

$$\Rightarrow a=\sqrt{yz},b=\sqrt{zx},c=\sqrt{xy}\therefore\sum_{\text{cyc}}\sqrt{xy}=3\to(2)\ (\because a+b+c=3)$$

$$\sqrt{\left(\frac{ab}{c}\right)^5}+\sqrt{\left(\frac{bc}{a}\right)^5}+\sqrt{\left(\frac{ca}{b}\right)^5}=\sum_{cyc}x^{\frac{5}{2}}\overset{?}{\geq}3\Leftrightarrow\sum_{cyc}u^5\underbrace{\overset{?}{\geq}}_{(*)}3\ (u=\sqrt{x},v=\sqrt{y},w=\sqrt{z})$$

$$\text{Now, }\sum_{cyc}u^5\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}u^4\right).\sqrt{\left(\sum_{cyc}u\right)^2}\geq\frac{1}{9}\left(\sum_{cyc}u^2\right)^2.\sqrt{3\sum_{cyc}uv}$$

$$\geq\frac{1}{9}\left(\sum_{cyc}uv\right)^2.\sqrt{3\sum_{cyc}uv}=\frac{1}{9}\left(\sum_{cyc}\sqrt{xy}\right)^2.\sqrt{3\sum_{cyc}\sqrt{xy}}\overset{\text{via }(2)}{=}\frac{1}{9}.9.\sqrt{3.3}=3$$

$$\Rightarrow(*)\text{ is true}\therefore\sqrt{\left(\frac{ab}{c}\right)^5}+\sqrt{\left(\frac{bc}{a}\right)^5}+\sqrt{\left(\frac{ca}{b}\right)^5}\geq3\overset{\text{via }(1)}{\geq}\frac{45}{2}.\frac{abc}{ab+bc+ca}-\frac{9}{2}$$

$$\Rightarrow\frac{9}{2}+\sqrt{\left(\frac{ab}{c}\right)^5}+\sqrt{\left(\frac{bc}{a}\right)^5}+\sqrt{\left(\frac{ca}{b}\right)^5}\geq\frac{45}{2}.\frac{abc}{ab+bc+ca},$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$

1474. If $a,b,c>0$, *then prove that* :

$$\sum_{cyc}\frac{1}{2(2a+b+c)}\geq\sum_{cyc}\frac{1}{3(a+b)+2c}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\Rightarrow a=s-x, b=s-y,$

$c=s-z$ and via such substitutions, $\sum_{cyc}\frac{1}{2(2a+b+c)}\geq\sum_{cyc}\frac{1}{3(a+b)+2c}$

$$\Leftrightarrow\sum_{cyc}\frac{1}{2(y+z)}\geq\sum_{cyc}\frac{1}{3x+2(s-x)}\Leftrightarrow\sum_{cyc}\frac{2s-x+x}{2s.2(2s-x)}\geq\sum_{cyc}\frac{2s+x-x}{2s(2s+x)}$$

$$\Leftrightarrow\frac{3}{4s}+\frac{1}{4s}\sum_{cyc}\frac{x}{2s-x}\geq\frac{3}{2s}-\frac{1}{2s}\sum_{cyc}\frac{x}{2s+x}\Leftrightarrow\sum_{cyc}\frac{x}{2s-x}+\sum_{cyc}\frac{2x}{2s+x}\geq3\rightarrow(1)$$

$$\text{Now, }\sum_{cyc}\frac{x}{2s-x}+\sum_{cyc}\frac{2x}{2s+x}=\sum_{cyc}\frac{x^2}{2sx-x^2}+2\sum_{cyc}\frac{x^2}{2sx+x^2}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right)^2 - \sum_{cyc} x^2} + \frac{2\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} x^2}$$

$$= \frac{m+2n}{m+2n-m} + \frac{2(m+2n)}{m+2n+m}\left(m = \sum_{cyc} x^2, n = \sum_{cyc} xy\right)$$

$$= \frac{(m+2n)(m+3n)}{2n(m+n)} \overset{?}{\geq} 3 \Leftrightarrow m^2 + 5mn + 6n^2 \overset{?}{\geq} 6mn + 6n^2 \Leftrightarrow m(m-n) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because m = \sum_{cyc} x^2 \geq \sum_{cyc} xy = n \Rightarrow (1) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{1}{2(2a+b+c)} \geq \sum_{cyc} \frac{1}{3(a+b)+2c} \forall a,b,c > 0, \text{"="} \text{ iff } a=b=c \text{ (QED)}$$

1475. In any Δ ABC, the following relationship holds :

$$(a^2+b^2+c^2)\sum_{cyc} \frac{1}{(a+b-c)^2} + \frac{81\prod_{cyc}(a+b-c)^2}{(ab+bc+ca)^3} \geq 12$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } \left(\sum_{cyc} ab\right)^2 \geq 3abc\sum_{cyc} a = 12Rrs.2s \Rightarrow \left(\sum_{cyc} ab\right)^2 \geq 24Rrs^2 \rightarrow (1)$$

$$\text{Now, } (a^2+b^2+c^2)\sum_{cyc} \frac{1}{(a+b-c)^2} + \frac{81\prod_{cyc}(a+b-c)^2}{(ab+bc+ca)^3} \geq 12$$

$$\Leftrightarrow \frac{1}{4}\left(\sum_{cyc} a^2\right)\frac{\left(\sum_{cyc}(s-b)(s-c)\right)^2 - 2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)}{r^4s^2}$$

$$+ \frac{81.64r^4s^2}{\left(\sum_{cyc} ab\right)^3} \geq 12$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^3.\left((4R+r)^2 - 2s^2\right) + 81.256r^6s^4 \overset{(*)}{\boxed{\geq}} 48r^2s^2\left(\sum_{cyc} ab\right)^3$$

$$\text{Now, via Trucht, (1) and} \because 3\left(\sum_{cyc} ab\right) \leq 4s^2, \text{LHS of } (*) - \text{RHS of } (*)$$

$$\geq \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right).24Rrs^4 + 81.256r^6s^4 - 16r^2s^2.4s^2\left(\sum_{cyc} ab\right)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3R(s^2+4Rr+r^2)(s^2-4Rr-r^2)+1296r^5-4r(s^2+4Rr+r^2)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (3R-4r)s^4-rs^2(32Rr+8r^2)-r^2(48R^3+88R^2r+35Rr^2-1292r^3) \boxed{\overset{?}{\underset{(**)}{\geq}}} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (3R-4r)(16Rr-5r^2)s^2-rs^2(32Rr+8r^2)$$

$$-r^2(48R^3+88R^2r+35Rr^2-1292r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow (48R^2-111Rr+12r^2)s^2-r(48R^3+88R^2r+35Rr^2-1292r^3) \boxed{\overset{?}{\underset{(***)}{\geq}}} 0$$

$$\boxed{\textit{Case 1}}\ 48R^2-111Rr+12r^2 \geq 0 \textit{ and then} : \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq}$$

$$(48R^2-111Rr+12r^2)(16Rr-5r^2)-r(48R^3+88R^2r+35Rr^2-1292r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow 90t^3-263t^2+89t+154 \overset{?}{\geq} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)((t-2)(90t+97)+117) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$$\boxed{\textit{Case 2}}\ 48R^2-111Rr+12r^2 < 0 \textit{ and then} : \text{LHS of } (***) =$$

$$-\left(-(48R^2-111Rr+12r^2)\right)s^2-r(48R^3+88R^2r+35Rr^2-1292r^3) \overset{\text{Gerretsen}}{\geq}$$

$$-\left(-(48R^2-111Rr+12r^2)\right)(4R^2+4Rr+3r^2)$$

$$-r(48R^3+88R^2r+35Rr^2-1292r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow 48t^4-75t^3-85t^2-80t+332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(48t^2+117t+191)+216\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \text{ is true} \therefore \text{combining both } \textit{cases}, (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (a^2+b^2+c^2)\sum_{cyc}\frac{1}{(a+b-c)^2}+\frac{81\prod_{cyc}(a+b-c)^2}{(ab+bc+ca)^3} \geq 12$$

$$\forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1476. If $a,b,c>0$ *such that* $\min\{a+b,b+c,c+a\} \geq 2$, then :

$$2\left(\sqrt[2]{1,25^{a+b+c}}-1\right) \geq \sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$(a+b)+(b+c)+(c+a) \geq 3\min\{a+b,b+c,c+a\} \geq 6 \Rightarrow \sum_{cyc} a \geq 3$$

$$\Rightarrow \frac{\sum_{cyc} a}{3} \geq 1 \Rightarrow 2\left(\sqrt[2]{1,25^{a+b+c}}-1\right) = 2\left(\left(\frac{5}{4}\right)^{\frac{\sum_{cyc} a}{3}}-1\right) \overset{\text{Bernoulli}}{\geq}$$

$$2\left(1+\frac{1}{4}\cdot\frac{\sum_{cyc}a}{3}-1\right)=\frac{\sum_{cyc}a}{6}\overset{A-G}{\geq}\frac{\sqrt[3]{abc}}{2}\overset{?}{\geq}\sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}$$

$$\Leftrightarrow (ab+1)(bc+1)(ca+1)\overset{?}{\geq}8abc \Leftrightarrow a^2b^2c^2+abc\sum_{cyc}a+\sum_{cyc}ab+1\underset{(*)}{\overset{?}{\geq}}8abc$$

$$\text{Now, LHS of }(*)\overset{A-G}{\geq}(abc)^2+3(abc)^{\frac{4}{3}}+3(abc)^{\frac{2}{3}}+1\overset{?}{\geq}8abc$$

$$\Leftrightarrow t^6+3t^4+3t^2+1\overset{?}{\geq}8t^3\ \left(t=\sqrt[3]{abc}\right)\Leftrightarrow t^6+3t^4-8t^3+3t^2+1\overset{?}{\geq}0$$

$$\Leftrightarrow (t-1)^2\left(t^4+2t^3+6t^2+2t+1\right)\overset{?}{\geq}0\to \text{true} \because t>0\Rightarrow(*)\text{ is true}$$

$$\therefore 2\left(\sqrt[2]{1,25^{a+b+c}}-1\right)\geq\sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}\ \forall a,b,c>0$$

$$\left|\min\{a+b,b+c,c+a\}\geq 2,''=''\text{ iff }a=b=c=1\text{ (QED)}\right.$$

1477. **If $a,b,c>0$ and $abc=1$, then prove that :**

$$\frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}$$

$$=\left(\frac{c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}}{b+c}\right)+\left(\frac{b^{2022}c^{2026}}{a+c}+\frac{(bc)^{2024}}{a+b}\right)+\left(\frac{a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}}{a+c}\right)$$

$$=c^{2022}a^{2024}\cdot\left(\frac{a^2}{a+b}+\frac{c^2}{b+c}\right)+b^{2022}c^{2024}\cdot\left(\frac{c^2}{c+a}+\frac{b^2}{a+b}\right)$$

$$+a^{2022}b^{2024}\cdot\left(\frac{b^2}{b+c}+\frac{a^2}{c+a}\right)$$

$$\overset{\text{Bergstrom}}{\geq}c^{2022}a^{2024}\cdot\frac{(c+a)^2}{(a+b)+(b+c)}+b^{2022}c^{2024}\cdot\frac{(b+c)^2}{(c+a)+(a+b)}$$

$$+a^{2022}b^{2024}\cdot\frac{(a+b)^2}{(b+c)+(c+a)}$$

$$\overset{A-G}{\geq}3\cdot\sqrt[3]{(abc)^{4046}\cdot\frac{(a+b)^2}{(b+c)+(c+a)}\cdot\frac{(b+c)^2}{(c+a)+(a+b)}\cdot\frac{(c+a)^2}{(a+b)+(b+c)}}$$

$$\overset{abc=1}{=} 3.\sqrt[3]{\frac{1}{abc}\cdot\frac{(a+b)^2}{(b+c)+(c+a)}\cdot\frac{(b+c)^2}{(c+a)+(a+b)}\cdot\frac{(c+a)^2}{(a+b)+(b+c)}}$$

$$\therefore \frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}$$

$$\geq 3.\sqrt[3]{\frac{1}{abc}\cdot\frac{(a+b)^2}{(b+c)+(c+a)}\cdot\frac{(b+c)^2}{(c+a)+(a+b)}\cdot\frac{(c+a)^2}{(a+b)+(b+c)}} \to (1)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=(s-x)(s-y)(s-z) \Rightarrow abc=r^2s$ and via such substitutions,

$$\text{RHS of } (1) \geq 3.\sqrt[3]{\frac{1}{r^2s}\cdot\frac{z^2}{x+y}\cdot\frac{x^2}{y+z}\cdot\frac{y^2}{z+x}} \overset{?}{\geq} 3 \Leftrightarrow \frac{16R^2r^2s^2}{r^2s.2s(s^2+2Rr+r^2)} \overset{?}{\geq} 1$$

$$\Leftrightarrow s^2 \overset{?}{\leq} 8R^2-2Rr-r^2 \Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r) \overset{?}{\leq} 0 \to \text{true}$$

$$\because s^2-4R^2-4Rr-3r^2 \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -2(2R+r)(R-2r) \overset{\text{Euler}}{\leq} 0$$

$$\Rightarrow \text{RHS of } (1) \geq 3 \overset{\text{via } (1)}{\Rightarrow}$$

$$\frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c} \geq 3$$

$\forall a,b,c>0 \mid abc=1,''='' \text{ iff } a=b=c=1$ (QED)

1478. If $a,b,c>0, abc=1$ then:

$$(a^{2024}+b^{2024})^{2025}+(b^{2024}+c^{2024})^{2025}+(c^{2024}+a^{2024})^{2025} \geq 3\cdot 2^{2025}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}(a^{2024}+b^{2024})^{2025} \overset{AM-GM}{\geq} \sum_{cyc}\left(2\sqrt{a^{2024}\cdot b^{2024}}\right)^{2025} =$$

$$= 2^{2025}\cdot\sum_{cyc}(a^{1012}\cdot b^{1012})^{2025} \overset{AM-GM}{\geq} 2^{2025}\cdot 3\sqrt[3]{\left(\prod_{cyc} a^{2024}\right)^{2025}} =$$

$$= 3\cdot 2^{2025}\cdot(abc)^{\frac{1}{3}\cdot 2024\cdot 2025} = 3\cdot 2^{2025}\cdot(1)^{\frac{1}{3}\cdot 2024\cdot 2025} = 3\cdot 2^{2025}$$

Equality holds for $a = b = c = 1$.

1479. If $a, b > 0$ and $a + b = 2$, then :

$$\frac{1}{a^a} + \frac{1}{b^b} \le 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x) = (2 - x).x^x - 1\ \forall\ x \in (0,1] \text{ and then :}$$
$$f'(x) = -x^x\big((x-2)\ln x + x - 1\big) = x^x.(2-x)\ln x - x^x.(x-1)$$
$$\le x^x.(2-x)(x-1) - x^x.(x-1) = x^x.(x-1)(2-x-1) = -x^x.(x-1)^2 \le 0$$
$$\Rightarrow f'(x) \le 0\ \forall\ x \in (0,1] \Rightarrow f(x) \text{ is } \downarrow \text{ on } (0,1] \Rightarrow f(x) \ge f(1) = 0\ \forall\ x \in (0,1]$$
$$\therefore (2-x).x^x - 1 \ge 0\ \forall\ x \in (0,1] \to (1)$$

$$\boxed{\text{Case 1}}\ 0 < a \le 1 \text{ and we have : } \frac{1}{b^b} = \frac{b}{b.b^b} = \frac{b^{1-b}}{b} = \frac{\big(1 + (b-1)\big)^{1-b}}{b}$$
$$\overset{\text{Bernoulli}}{\le} \frac{1 + (b-1)(1-b)}{b} = 2 - b \Rightarrow \frac{1}{a^a} + \frac{1}{b^b} \le \frac{1}{a^a} + 2 - b \overset{?}{\le} 2 \Leftrightarrow \frac{1}{a^a} \overset{?}{\le} b$$
$$= 2 - a \Leftrightarrow (2-a).a^a - 1 \overset{?}{\ge} 0 \to \text{true via } (1) \therefore \frac{1}{a^a} + \frac{1}{b^b} \le 2$$

$$\boxed{\text{Case 2}}\ 1 \le a < 2 \text{ and then : } 0 < b \le 1 \text{ and we have : } \frac{1}{a^a} = \frac{a}{a.a^a} = \frac{a^{1-a}}{a}$$
$$= \frac{\big(1 + (a-1)\big)^{1-a}}{a} \overset{\text{Bernoulli}}{\le} \frac{1 + (a-1)(1-a)}{a} = 2 - a$$
$$\Rightarrow \frac{1}{a^a} + \frac{1}{b^b} \le 2 - a + \frac{1}{b^b} \overset{?}{\le} 2 \Leftrightarrow \frac{1}{b^b} \overset{?}{\le} a = 2 - b \Leftrightarrow (2-b).b^b - 1 \overset{?}{\ge} 0$$
$$\to \text{true via } (1) \therefore \frac{1}{a^a} + \frac{1}{b^b} \le 2 \therefore \text{combining cases } (1) \text{ and } (2), \frac{1}{a^a} + \frac{1}{b^b} \le 2$$
$$\forall\ a, b > 0 \,\big|\, a + b = 2, \text{"="} \text{ iff } a = b = 1 \text{ (QED)}$$

1480. If $a, b > 0$ then:

$$\frac{1}{1+2a} + \frac{1}{1+2b} \ge \frac{2}{ab+2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{1+2a} + \frac{1}{1+2b} \ge \frac{2}{ab+2} \Leftrightarrow \frac{1 + 2a + 1 + 2b}{(1+2a)(1+2b)} \ge \frac{2}{ab+2}$$

$$\frac{2(1+a+b)}{1 + 2(a+b) + 4ab} \ge \frac{2}{ab+2}, \qquad S = a + b, P = ab$$

$$\frac{1+S}{1+2S+4P} \ge \frac{1}{P+2} \Leftrightarrow (1+S)(P+2) \ge 1+2S+4P$$

$$P+2+SP+2S \ge 1+2S+4P, \quad SP+1 \ge 3P \text{ (to prove)}$$

$$SP+1 \overset{AM-GM}{\ge} 2\sqrt{P}\cdot P+1 \ge 3P \text{ (to prove)}, \sqrt{P}=x \text{ (denote)}$$

$$2x^3+1 \ge 3x^2 \Leftrightarrow 2x^3+1-3x^2 \ge 0$$

$$2x^3-2x^2-x^2+1 \ge 0 \Leftrightarrow 2x^2(x-1)-(x-1)(x+1) \ge 0$$

$$(x-1)(2x^2-x-1) \ge 0 \Leftrightarrow (x-1)^2(2x+1) \ge 0$$

Equality holds for $a=b=1$.

1481. If $a, b > 0$, then:

$$\sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \le \sqrt{\frac{8}{a+b}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \overset{CBS}{\le} \sqrt{a+2b+b+2a}.\sqrt{\frac{1}{a^2+2b^2}+\frac{1}{b^2+2a^2}}$$

$$\overset{?}{\le} \sqrt{\frac{8}{a+b}} \Leftrightarrow 3(a+b).\sqrt{a^2+b^2} \overset{?}{\le} \sqrt{8(a^2+2b^2)(b^2+2a^2)}$$

$$\Leftrightarrow 8(a^2+2b^2)(b^2+2a^2) \overset{?}{\ge} 9(a^2+b^2)(a+b)^2$$

$$\Leftrightarrow 7t^4-18t^3+22t^2-18t+7 \overset{?}{\ge} 0 \left(t=\frac{a}{b}\right) \Leftrightarrow (t-1)^2(7(t-1)^2+10t) \overset{?}{\ge} 0$$

$$\to \text{true} \because t>0 \therefore \sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \le \sqrt{\frac{8}{a+b}}, \text{''}=\text{''} \text{ iff } a=b \text{ (QED)}$$

1482. If $a, b > 0$ and $a+b=2$

then: $a^{2b}+b^{2a} \le 2$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^{2b} = \left(\left(1+(a-1)\right)^{\frac{b}{2}}\right)^4 \overset{\text{Bernoulli} \because \frac{b}{2}<1}{\leq} \left(1+\frac{b(a-1)}{2}\right)^4$$

$$= \frac{1}{16}(2-b+ab)^4 \overset{a+b=2}{=} \frac{1}{16}(a+ab)^4 = \frac{1}{16}.a^4(1+b)^4 \overset{a+b=2}{=} \frac{1}{16}.(2-b)^4(1+b)^4$$

$$\therefore a^{2b} \leq \frac{1}{16}.(2-b)^4(1+b)^4 \rightarrow (1)$$

$$\text{Again, } b^{2a} = \left(\left(1+(b-1)\right)^{\frac{a}{2}}\right)^4 \overset{\text{Bernoulli} \because \frac{a}{2}<1}{\leq} \left(1+\frac{a(b-1)}{2}\right)^4$$

$$= \frac{1}{16}(2-a+ab)^4 \overset{a+b=2}{=} \frac{1}{16}(b+ab)^4 = \frac{1}{16}.b^4(1+a)^4 \overset{a+b=2}{=} \frac{1}{16}.b^4(3-b)^4$$

$$\therefore b^{2a} \leq \frac{1}{16}.b^4(3-b)^4 \rightarrow (2) \therefore (1),(2) \Rightarrow a^{2b}+b^{2a}$$

$$\leq \frac{1}{16}.(2-b)^4(1+b)^4 + \frac{1}{16}.b^4(3-b)^4 \overset{?}{\leq} 2$$

$$\Leftrightarrow b^8-8b^7+26b^6-44b^5+41b^4-20b^3-4b^2+16b-8 \underset{(*)}{\overset{?}{\leq}} 0$$

$$\Leftrightarrow (b-1)^2\left(b^6-6b^5+13b^4-12b^3+4b^2-8\right) \overset{?}{\leq} 0 \text{ and to prove this,}$$

$$\text{it suffices to prove : } b^6-6b^5+13b^4-12b^3+4b^2<8$$

$$\Leftrightarrow \boxed{b^2(b-1)^2(b-2)^2 \overset{(**)}{<} 8}$$

$$\text{Now,} \because 0<b<2 \therefore -1<b-1<1 \Rightarrow |b-1|<1 \Rightarrow \boxed{(b-1)^2<1} \rightarrow (i)$$

$$\text{Also, } 2 = a+b \overset{A-G}{\geq} 2\sqrt{ab} \Rightarrow ab \leq 1 \Rightarrow b^2(b-2)^2 = b^2(2-b)^2 \overset{a+b=2}{=} a^2b^2 \leq 1$$

$$\Rightarrow \boxed{b^2(b-2)^2 \leq 1} \rightarrow (ii)$$

$$\therefore (i)\bullet(ii) \Rightarrow b^2(b-1)^2(b-2)^2 < 1 < 8 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore a^{2b}+b^{2a} \leq 2 \ \forall a,b>0 \mid a+b=2, " = " \text{ iff } a=b=1 \text{ (QED)}$$

1483. If $a, b > 0$ and $a+b = a^4+b^4$

then : $a^a b^b \leq 1$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via weighted AM}-\text{GM inequality}, \sqrt[a+b]{(a^3)^a.(b^3)^b} \le \frac{a^3.a+b^3.b}{a+b}$$
$$= \frac{a^4+b^4}{a+b} = 1 \Rightarrow \frac{1}{a+b}.\ln x \le 0 \left(x = (a^3)^a.(b^3)^b\right) \Rightarrow \ln x \le 0$$
$$\Rightarrow x = (a^3)^a.(b^3)^b \le 1 \Rightarrow (a^a)^3.(b^b)^3 \le 1$$
$$\Rightarrow (a^a b^b)^3 \le 1 \Rightarrow a^a b^b \le 1\ \forall a,b>0 \mid a+b = a^4+b^4, '' = ''\ \text{iff } a=b=1\ (\text{QED})$$

1484. If $a, b > 0$ and $a^9 + b^9 = 2$

$$\text{then}: \frac{a^2}{b} + \frac{b^2}{a} \ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^9+b^9 = 2 \Rightarrow (a^3+b^3)(a^6+b^6-a^3b^3) = 2$$
$$\Rightarrow (a^3+b^3)\left((a^3+b^3)^2 - 3a^3b^3\right) = 2 \Rightarrow (a^3+b^3)^3 - 2 = 3a^3b^3(a^3+b^3)$$
$$\Rightarrow a^3b^3 = \frac{(a^3+b^3)^3-2}{3(a^3+b^3)} \to (1)$$
$$\text{Now}, \frac{a^2}{b}+\frac{b^2}{a} \ge 2 \Leftrightarrow a^3+b^3 \ge 2ab \Leftrightarrow \frac{(a^3+b^3)^3}{8} \ge a^3b^3$$
$$\overset{\text{via (1)}}{\Leftrightarrow} \frac{(a^3+b^3)^3}{8} \ge \frac{(a^3+b^3)^3-2}{3(a^3+b^3)} \Leftrightarrow 3t^4 \ge 8(t^3-2)\ (t = a^3+b^3)$$
$$\Leftrightarrow 3t^4 - 8t^3 + 16 \ge 0 \Leftrightarrow (t-2)^2(3t^2+4t+4) \ge 0 \to \text{true}$$
$$\because t > 0 \therefore \frac{a^2}{b}+\frac{b^2}{a} \ge 2\ \forall a,b>0 \mid a^9+b^9 = 2, '' = ''\ \text{iff } a=b=1\ (\text{QED})$$

1485.

If $a, b, c > 0$ and $a^2 + b^2 + c^2 = a + b + c$, then:
$$a^2b^2 + b^2c^2 + c^2a^2 \le ab + bc + ca$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$$\therefore abc = r^2s \to (2) \text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4),$$

$$\text{and} \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

$$a^2b^2 + b^2c^2 + c^2a^2 \le ab + bc + ca \overset{a^2+b^2+c^2 = a+b+c}{\Leftrightarrow}$$

$$\sum_{cyc} a^2b^2 \le \left(\sum_{cyc} ab\right)\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} a}\right)^2 \overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow} (4Rr + r^2)(s^2 - 8Rr - 2r^2)^2$$

$$\ge s^2r^2\left((4R + r)^2 - 2s^2\right) \Leftrightarrow (4R + 3r)s^4 - 5rs^2(4R + r)^2 + 4r^2(4R + r)^3 \overset{(*)}{\ge} 0$$

$$\text{and} \because (4R + 3r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \ge (4R + 3r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (48R^2 + 16Rr - 35r^2)s^2 \overset{(**)}{\ge} r(768R^3 - 64R^2r - 428Rr^2 + 71r^3)$$

$$\text{Now, } (48R^2 + 16Rr - 35r^2)s^2 \overset{\text{Gerretsen}}{\ge} (48R^2 + 16Rr - 35r^2)(16Rr - 5r^2)$$

$$\overset{?}{\ge} r(768R^3 - 64R^2r - 428Rr^2 + 71r^3) \Leftrightarrow 20R^2 - 53Rr + 26r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R - 2r)(20R - 13r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore a^2b^2 + b^2c^2 + c^2a^2 \le ab + bc + ca$$

$$\forall a,b,c > 0 \,\middle|\, a^2 + b^2 + c^2 = a + b + c, \text{''}=\text{''} \text{ iff } a = b = c = 1 \text{ (QED)}$$

1486. If $a,b,c > 0$ and $abc = 1$, then prove that :

$$\frac{a^7c^2 + b^9}{ab(a^3 + b^3)} + \frac{b^7a^2 + c^9}{bc(b^3 + c^3)} + \frac{c^7b^2 + a^9}{ac(a^3 + c^3)} \ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z} \text{ and then : } \frac{a^7c^2}{ab(a^3 + b^3)} \overset{\frac{1}{ab} = xy}{=} \frac{\frac{1}{x^7z^2}.xy}{\frac{1}{x^3} + \frac{1}{y^3}}$$

$$=\frac{\frac{1}{x^4}\cdot\frac{y}{x^2z^2}}{\frac{1}{x^3}+\frac{1}{y^3}}\overset{y=\frac{1}{zx}}{=}\frac{\frac{1}{x^4}\cdot\frac{1}{x^3z^3}}{\frac{x^3+y^3}{x^3y^3}}=\frac{\frac{1}{x^4}\cdot\frac{y^3}{z^3}}{x^3+y^3}=\frac{\frac{1}{x^4}\cdot\left(\frac{y}{z}\right)^4}{x^3.\frac{y}{z}+y^3.\frac{y}{z}}\overset{\frac{1}{zx}=y}{=}\frac{y^4.y^4}{x^3y.xy+y^4.xy}$$

$$=\frac{y^8}{xy^2(x^3+y^3)}\Rightarrow\frac{a^7c^2}{ab(a^3+b^3)}=\frac{y^6}{x(x^3+y^3)}\text{ and analogously,}$$

$$\frac{b^7a^2}{bc(b^3+c^3)}=\frac{z^6}{y(y^3+z^3)}\text{ and }\frac{c^7b^2}{ac(a^3+c^3)}=\frac{x^6}{z(z^3+x^3)}$$

$$\therefore\frac{a^7c^2}{ab(a^3+b^3)}+\frac{b^7a^2}{bc(b^3+c^3)}+\frac{c^7b^2}{ac(a^3+c^3)}$$

$$=\frac{y^6}{x(x^3+y^3)}+\frac{z^6}{y(y^3+z^3)}+\frac{x^6}{z(z^3+x^3)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^3\right)^2}{\sum_{cyc}\left(x\left(\sum_{cyc}x^3-z^3\right)\right)}$$

$$=\frac{\left(\sum_{cyc}x^3\right)^2}{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^3\right)-\sum_{cyc}x^3y}\geq\frac{\left(\sum_{cyc}x^3\right)^2}{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^3\right)-3}$$

$$\left(\because\sum_{cyc}x^3y\overset{A-G}{\geq}3.\sqrt[3]{(xyz)^4}\overset{xyz=1}{=}3\right)\geq\frac{\left(\sum_{cyc}x^3\right)^2}{\sqrt[3]{9\sum_{cyc}x^3}.\left(\sum_{cyc}x^3\right)-3}$$

$$\left(\because\sum_{cyc}x^3\overset{\text{Holder}}{\geq}\frac{1}{9}\left(\sum_{cyc}x\right)^3\Rightarrow\sum_{cyc}x\leq\sqrt[3]{9\sum_{cyc}x^3}\right)=\frac{t^2}{t.\sqrt[3]{9t}-3}\left(t=\sum_{cyc}x^3\right)\overset{?}{\geq}\frac{3}{2}$$

$$\Leftrightarrow 2t^2+9\overset{?}{\geq}3t.\sqrt[3]{9t}\Leftrightarrow\left(2t^2+9\right)^3\overset{?}{\geq}243t^4\Leftrightarrow 8t^6-135t^4+486t^2+729\overset{?}{\geq}0$$

$$\Leftrightarrow(t-3)^2\left(8t^4+48t^3+81t^2+54t+81\right)\overset{?}{\geq}0\to\text{true,}$$

$$''=''\text{ iff }t=\sum_{cyc}x^3=3\text{ and }\because\sum_{cyc}x^3\overset{A-G}{\geq}3xyz\overset{xyz=1}{=}3\ \therefore''=''\text{ iff }x=y=z=1$$

$$\therefore\frac{a^7c^2}{ab(a^3+b^3)}+\frac{b^7a^2}{bc(b^3+c^3)}+\frac{c^7b^2}{ac(a^3+c^3)}\geq\frac{3}{2},''=''\text{ iff }a=b=c=1\to(1)$$

$$\text{Now, }a^4+b^4+b^4+b^4\overset{A-G}{\geq}4ab^3,b^4+c^4+c^4+c^4\overset{A-G}{\geq}4bc^3,$$

$$c^4+a^4+a^4+a^4\overset{A-G}{\geq}4ca^3\ \therefore 4\sum_{cyc}a^4\geq 4\sum_{cyc}ab^3\Rightarrow\sum_{cyc}ab^3\leq\sum_{cyc}a^4\to(\text{i})$$

$$\text{We have}:\frac{b^9}{ab(a^3+b^3)}+\frac{c^9}{bc(b^3+c^3)}+\frac{a^9}{ac(a^3+c^3)}$$

$$=\frac{b^8}{a^4+ab^3}+\frac{c^8}{b^4+bc^3}+\frac{a^8}{c^4+ca^3}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^4\right)^2}{\sum_{cyc}a^4+\sum_{cyc}ab^3}\overset{\text{via (i)}}{\geq}\frac{\left(\sum_{cyc}a^4\right)^2}{2\sum_{cyc}a^4}$$

$$=\frac{1}{2}\sum_{cyc}a^4\overset{A-G}{\geq}\frac{3}{2}.\sqrt[3]{(abc)^4}\overset{abc=1}{=}\frac{3}{2}$$

$$\Rightarrow\frac{b^9}{ab(a^3+b^3)}+\frac{c^9}{bc(b^3+c^3)}+\frac{a^9}{ac(a^3+c^3)}\geq\frac{3}{2},''=''\text{ iff }a=b=c=1\to(2)$$

$$\therefore(1)+(2)\Rightarrow\frac{a^7c^2+b^9}{ab(a^3+b^3)}+\frac{b^7a^2+c^9}{bc(b^3+c^3)}+\frac{c^7b^2+a^9}{ac(a^3+c^3)}\geq 3,$$

$'' ='' \text{ iff } a = b = c = 1$ (QED)

1487. If $a, b > 0$, *then prove that* :

$$(a+b)\left(\frac{1}{\sqrt{a^2-ab+2b^2}}+\frac{1}{\sqrt{b^2-ab+2a^2}}\right)\le 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a+b)\left(\frac{1}{\sqrt{a^2-ab+2b^2}}+\frac{1}{\sqrt{b^2-ab+2a^2}}\right)$$

$$=(a+b)\left(\frac{1}{\sqrt{\sqrt{a^2-ab+b^2}^2+b^2}}+\frac{1}{\sqrt{\sqrt{a^2-ab+b^2}^2+a^2}}\right)$$

$$\le(a+b)\left(\frac{1}{\sqrt{\frac{1}{2}\left(\sqrt{a^2-ab+b^2}+b\right)^2}}+\frac{1}{\sqrt{\frac{1}{2}\left(\sqrt{a^2-ab+b^2}+a\right)^2}}\right)$$

$$=\sqrt{2}(a+b)\left(\frac{1}{\sqrt{a^2-ab+b^2}+b}+\frac{1}{\sqrt{a^2-ab+b^2}+a}\right)$$

$$=\frac{\sqrt{2}(a+b)\left(a+b+2\sqrt{a^2-ab+b^2}\right)}{a^2-ab+b^2+ab+(a+b).\sqrt{a^2-ab+b^2}}$$

$$=\frac{2\sqrt{2}\left((a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}\right)}{2(a^2+b^2)+2(a+b).\sqrt{a^2-ab+b^2}}$$

$$\le\frac{2\sqrt{2}\left((a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}\right)}{(a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}}=2\sqrt{2},''='' \text{ iff } a=b \text{ (QED)}$$

1488. If $a, b, c > 0$ and $abc(a+b+c)^3 = 27$, then prove that :

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\ge 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\ge\prod_{cyc}\left(\frac{1}{2}(a+b)^2\right)\overset{?}{\ge}8\Leftrightarrow\prod_{cyc}(a+b)^2\overset{?}{\ge}64$$

$$=\frac{64}{27}.27\overset{abc(a+b+c)^3=27}{=}\frac{64}{27}.abc(a+b+c)^3\Leftrightarrow 27\prod_{cyc}(a+b)^2\underset{(*)}{\overset{?}{\geq}}64abc\left(\sum_{cyc}a\right)^3$$

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x,b=s-y,c=s-z$

$$\therefore abc=\prod_{cyc}(s-x)\Rightarrow abc=r^2s\to(2)\therefore(1),(2)\Rightarrow(*)\Leftrightarrow 27x^2y^2z^2\geq 64r^2s.s^3$$

$$\Leftrightarrow 27.16R^2r^2s^2\geq 64r^2s^4\Leftrightarrow 27R^2\geq 4s^2\Leftrightarrow s\leq\frac{3\sqrt{3}R}{2}\to\text{true via Mitrinovic}$$

$$\Rightarrow(*)\text{ is true}\therefore(a^2+b^2)(b^2+c^2)(c^2+a^2)\geq 8$$

$$\forall a,b,c>0\,\big|\,abc(a+b+c)^3=27,''='' \text{ iff } a=b=c=1\text{ (QED)}$$

1489. If $a,b,c>0$ and $(a+b)(b+c)(c+a)=1$, then prove that :

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}\geq\frac{4}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}$$

$$=\frac{a^2}{ab(b+2c)^2}+\frac{b^2}{bc(c+2a)^2}+\frac{c^2}{ca(a+2b)^2}$$

$$=\frac{\left(\frac{a}{b+2c}\right)^2}{ab}+\frac{\left(\frac{b}{c+2a}\right)^2}{bc}$$

$$+\frac{\left(\frac{c}{a+2b}\right)^2}{ca}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{a^2}{ab+2ca}\right)^2}{\sum_{cyc}ab}\overset{\text{Bergstrom}}{\geq}\frac{\left(\frac{\left(\sum_{cyc}a\right)^2}{3\sum_{cyc}ab}\right)^2}{\sum_{cyc}ab}$$

$$\overset{\left(\sum_{cyc}a\right)^2\geq 3\sum_{cyc}ab}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{3\left(\sum_{cyc}ab\right)^2}\overset{?}{\geq}\frac{4}{3}\Leftrightarrow\sum_{cyc}a\overset{?}{\geq}2\sum_{cyc}ab$$

$$\overset{(a+b)(b+c)(c+a)=1}{\Leftrightarrow}\left(\sum_{cyc}a\right).\sqrt[3]{(a+b)(b+c)(c+a)}\underset{(*)}{\overset{?}{\geq}}2\sum_{cyc}ab$$

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a$

> 0 and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$$\therefore \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) = 4Rr + r^2 \rightarrow (2) \therefore\rightarrow (1),(2) \Rightarrow (*) \Leftrightarrow$$

$$s.\sqrt[3]{xyz} \geq 2(4Rr + r^2) \Leftrightarrow s^3.4Rs \geq 8r^3(4R+r)^3 \Leftrightarrow Rs^4 \overset{(**)}{\geq} 2r^2(4R+r)^3$$

$$\text{Now, } s^2 \overset{\text{Gerretsen}}{\geq} 16Rr - 5r^2 = 3r(4R+r) + 4r(R-2r) \overset{\text{Euler}}{\geq} 3r(4R+r)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : $R\big(3r(4R+r)\big)^2 \geq 2r^2(4R+r)^3$

$\Leftrightarrow 9R \geq 2(4R+r) \Leftrightarrow R \geq 2r \rightarrow$ true via Euler $\Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore \frac{a}{b(b+2c)^2} + \frac{b}{c(c+2a)^2} + \frac{c}{a(a+2b)^2} \geq \frac{4}{3}$$

$$\forall a, b, c > 0 \,\big|\, (a+b)(b+c)(c+a) = 1, \text{"="} \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$

1490. If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a^2b^2(b^7c^2 + a^9)}{c(a^5+b^5)} + \frac{b^2c^2(c^7a^2 + b^9)}{a(b^5+c^5)} + \frac{c^2a^2(a^7b^2 + c^9)}{b(a^5+c^5)} \geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$3b^5 + 2a^5 \overset{A-G}{\geq} 5b^3a^2, 3c^5 + 2b^5 \overset{A-G}{\geq} 5c^3b^2, 3a^5 + 2c^5 \overset{A-G}{\geq} 5a^3c^2$$

$$\therefore \text{via summation, } 5\sum_{cyc} a^5 \geq 5\sum_{cyc} a^2b^3 \Rightarrow \sum_{cyc} a^2b^3 \leq \sum_{cyc} a^5 \rightarrow (1)$$

$$\text{Now, } \frac{a^2b^2(b^7c^2 + a^9)}{c(a^5+b^5)} + \frac{b^2c^2(c^7a^2 + b^9)}{a(b^5+c^5)} + \frac{c^2a^2(a^7b^2 + c^9)}{b(a^5+c^5)} \overset{abc=1}{=}$$

$$\frac{b^7}{c(a^5+b^5)} + \frac{a^9}{c^3(a^5+b^5)} + \frac{c^7}{a(b^5+c^5)} + \frac{b^9}{a^3(b^5+c^5)} + \frac{a^7}{b(a^5+c^5)} + \frac{c^9}{b^3(a^5+c^5)}$$

$$= \frac{b^{10}}{b^3c(a^5+b^5)} + \frac{a^{10}}{c^3a(a^5+b^5)} + \frac{c^{10}}{c^3a(b^5+c^5)} + \frac{b^{10}}{a^3b(b^5+c^5)} + \frac{a^{10}}{a^3b(a^5+c^5)} + \frac{c^{10}}{b^3c(a^5+c^5)}$$

$$= a^{10}.\left(\frac{1}{c^3a(a^5+b^5)} + \frac{1}{a^3b(a^5+c^5)}\right) + b^{10}.\left(\frac{1}{b^3c(a^5+b^5)} + \frac{1}{a^3b(b^5+c^5)}\right)$$

$$+c^{10}.\left(\frac{1}{c^3a(b^5+c^5)} + \frac{1}{b^3c(a^5+c^5)}\right) \overset{\substack{A-G \\ \text{and} \\ \because abc=1}}{\geq}$$

$$\frac{2a^{10}}{\sqrt{a^3c^2(c^5+a^5)(a^5+b^5)}} + \frac{2b^{10}}{\sqrt{b^3a^2(a^5+b^5)(b^5+c^5)}} + \frac{2c^{10}}{\sqrt{c^3b^2(b^5+c^5)(c^5+a^5)}}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{2\left(\sum_{cyc} a^5\right)^2}{\sum_{cyc}\sqrt{a^2b^3(a^5+b^5)(b^5+c^5)}} \overset{\text{CBS}}{\geq} \frac{2\left(\sum_{cyc} a^5\right)^2}{\sqrt{\sum_{cyc} a^2b^3 . \sum_{cyc}\sqrt{(a^5+b^5)(b^5+c^5)}}}$$

$$\overset{\substack{\text{via (1)} \\ \text{and} \\ \because 3\sum_{cyc} xy \leq \left(\sum_{cyc} x\right)^2}}{\geq} \frac{2\left(\sum_{cyc} a^5\right)^2}{\sqrt{\sum_{cyc} a^5 . \sum_{cyc}\sqrt{\frac{\left(\sum_{cyc}(b^5+c^5)\right)^2}{3}}}} = \frac{2\sqrt{3}.\left(\sum_{cyc} a^5\right)^2}{\sqrt{\sum_{cyc} a^5 . \left(2\sum_{cyc} a^5\right)}}$$

$$= \sqrt{3\sum_{cyc} a^5} \overset{A-G}{\geq} \sqrt{9(abc)^{\frac{5}{3}}} \overset{abc=1}{=} 3$$

$$\therefore \frac{a^2b^2(b^7c^2+a^9)}{c(a^5+b^5)} + \frac{b^2c^2(c^7a^2+b^9)}{a(b^5+c^5)} + \frac{c^2a^2(a^7b^2+c^9)}{b(a^5+c^5)} \geq 3$$

$$\forall\, a,b,c > 0 \mid abc = 1, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1491. If $a,b,c > 0$ and $abc = 1$, then $\forall\, m,n \in \mathbb{N}$, prove that :

$$\frac{b^{2m}.\sqrt{b^n+c^n}+a^{2m}.\sqrt{a^n+c^n}}{a^m+b^m} + \frac{b^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{a^n+c^n}}{b^m+c^m}$$

$$+\frac{a^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{b^n+c^n}}{a^m+c^m} \geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = \sqrt{a^n+b^n}.\left(\frac{b^{2m}}{b^m+c^m}+\frac{a^{2m}}{c^m+a^m}\right)$$

$$+\sqrt{b^n+c^n}.\left(\frac{b^{2m}}{a^m+b^m}+\frac{c^{2m}}{c^m+a^m}\right)+\sqrt{c^n+a^n}.\left(\frac{a^{2m}}{a^m+b^m}+\frac{c^{2m}}{b^m+c^m}\right)$$

$$\overset{\text{Bergstrom}}{\geq} \sqrt{a^n+b^n}.\frac{(a^m+b^m)^2}{\left((b^m+c^m)+(c^m+a^m)\right)}$$

$$+\sqrt{b^n+c^n}.\frac{(b^m+c^m)^2}{\left((c^m+a^m)+(a^m+b^m)\right)}+\sqrt{c^n+a^n}.\frac{(c^m+a^m)^2}{\left((a^m+b^m)+(b^m+c^m)\right)} \overset{A-G}{\geq}$$

$$3.\sqrt[3]{\sqrt{\prod_{cyc}(b^n+c^n)}.\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{\prod_{cyc}\left((b^m+c^m)+(c^m+a^m)\right)}} \overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[3]{\sqrt{8a^n b^n c^n}.\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{((b^m+c^m)+(c^m+a^m))((c^m+a^m)+(a^m+b^m))((a^m+b^m)+(b^m+c^m))}}$$

$$\overset{abc=1}{=}3\sqrt{2}.\sqrt[3]{\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{a^m b^m c^m.\prod_{cyc}((b^m+c^m)+(c^m+a^m))}}\overset{?}{\geq}3\sqrt{2}$$

$$\Leftrightarrow\sqrt[3]{\frac{1}{xyz}.\frac{(x+y)^2.(y+z)^2.(z+x)^2}{((y+z)+(z+x))((z+x)+(x+y))((x+y)+(y+z))}}\overset{?}{\geq}1\to(1)$$

$$(x=a^m, y=b^m, z=c^m)$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\Rightarrow x=s-X, y=s-Y, z=s-Z$ and such substitutions $\Rightarrow xyz=(s-X)(s-Y)(s-Z)\Rightarrow xyz=r^2s\therefore$ LHS of (1)

$$\geq\sqrt[3]{\frac{1}{r^2s}.\frac{Z^2X^2Y^2}{(X+Y)(Y+Z)(Z+X)}}\overset{?}{\geq}1\Leftrightarrow\frac{16R^2r^2s^2}{r^2s.2s(s^2+2Rr+r^2)}\overset{?}{\geq}1$$

$$\Leftrightarrow s^2\overset{?}{\leq}8R^2-2Rr-r^2\Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r)\overset{?}{\leq}0\to\text{true}$$

$$\because s^2-4R^2-4Rr-3r^2\overset{\text{Gerretsen}}{\leq}0\text{ and }-2(2R+r)(R-2r)\overset{\text{Euler}}{\leq}0\Rightarrow(1)\text{ is true}$$

$$\Rightarrow\frac{b^{2m}.\sqrt{b^n+c^n}+a^{2m}.\sqrt{a^n+c^n}}{a^m+b^m}+\frac{b^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{a^n+c^n}}{b^m+c^m}$$

$$+\frac{a^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{b^n+c^n}}{a^m+c^m}\geq3\sqrt{2}\ \forall\, a,b,c>0\,|\,abc=1$$

and $\forall\, m,n\in\mathbb{N}$, "=" iff $a=b=c=1$ (QED)

1492. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{\sqrt{a+b}.(b^2.\sqrt{bc}+a^2.\sqrt{ac})}{b.\sqrt{b+c}+a.\sqrt{a+c}}+\frac{\sqrt{b+c}.(b^2.\sqrt{ab}+c^2.\sqrt{ac})}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.(a^2.\sqrt{ab}+c^2.\sqrt{bc})}{a.\sqrt{a+b}+c.\sqrt{b+c}}\geq3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}} \overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{CBS}}}{\geq} \frac{\sqrt{a+b}.\frac{1}{2}(b^2+a^2).\sqrt{c}.\left(\sqrt{b}+\sqrt{a}\right)}{\sqrt{b^2+a^2}.\sqrt{b+c+c+a}}$$

$$\overset{abc=1}{=} \frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\right)}{\sqrt{b+c+c+a}} = \frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)}{\sqrt{b+c+c+a}}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{4}{\sqrt{a}+\sqrt{b}}\right)}{\sqrt{b+c+c+a}} \overset{\text{CBS}}{\geq} \frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{4}{\sqrt{2(a+b)}}\right)}{\sqrt{b+c+c+a}}$$

$$= \frac{\sqrt{2(a^2+b^2)}}{\sqrt{b+c+c+a}} \overset{\text{Reverse CBS}}{\geq} \frac{a+b}{\sqrt{b+c+c+a}}$$

$$\therefore \frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}} \geq \frac{a+b}{\sqrt{b+c+c+a}} \text{ and analogs}$$

$$\Rightarrow \frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}} + \frac{\sqrt{b+c}.\left(b^2.\sqrt{ab}+c^2.\sqrt{ac}\right)}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.\left(a^2.\sqrt{ab}+c^2.\sqrt{bc}\right)}{a.\sqrt{a+b}+c.\sqrt{b+c}} \geq \sum_{\text{cyc}} \frac{a+b}{\sqrt{b+c+c+a}} \overset{A-G}{\geq} 3.\sqrt[3]{\frac{\prod_{\text{cyc}}(a+b)}{\prod_{\text{cyc}}\sqrt{b+c+c+a}}}$$

$$\overset{abc=1}{=} 3.\sqrt[3]{\frac{\prod_{\text{cyc}}(a+b)}{\sqrt{abc}.\prod_{\text{cyc}}\sqrt{b+c+c+a}}} \overset{?}{\geq} 3 \Leftrightarrow \prod_{\text{cyc}}(a+b)^2 \underset{(*)}{\overset{?}{\geq}} abc\prod_{\text{cyc}}(b+c+c+a)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = (s-x)(s-y)(s-z) \Rightarrow abc = r^2s$ and via such substitutions, $(*) \Leftrightarrow$

$$x^2y^2z^2 \geq r^2s(x+y)(y+z)(z+x) \Leftrightarrow 16R^2r^2s^2 \geq r^2s.2s(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^2 \leq 8R^2-2Rr-r^2 \Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r) \leq 0 \rightarrow \text{true}$$

$$\because s^2-4R^2-4Rr-3r^2 \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -2(2R+r)(R-2r) \overset{\text{Euler}}{\leq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}} + \frac{\sqrt{b+c}.\left(b^2.\sqrt{ab}+c^2.\sqrt{ac}\right)}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.\left(a^2.\sqrt{ab}+c^2.\sqrt{bc}\right)}{a.\sqrt{a+b}+c.\sqrt{b+c}} \geq 3 \ \forall\, a,b,c>0 \,|\, abc=1,$$

$''=''$ iff $a=b=c=1$ (QED)

1493. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a^2b^2.\sqrt{ab}(\sqrt{b+c}+\sqrt{c+a})}{b^2c^2.\sqrt{bc}+c^2a^2.\sqrt{ca}}+\frac{b^2c^2.\sqrt{bc}(\sqrt{a+b}+\sqrt{c+a})}{a^2b^2.\sqrt{ab}+c^2a^2.\sqrt{ca}}$$
$$+\frac{c^2a^2.\sqrt{ca}(\sqrt{a+b}+\sqrt{b+c})}{a^2b^2.\sqrt{ab}+b^2c^2.\sqrt{bc}}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\ A,B,C>0,(A+B),(B+C),(C+A)\text{ form sides of } a \text{ triangle}$$
$$(\because (A+B)+(B+C)>(C+A)\text{ and analogs})$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}\text{ form sides of } a \text{ triangle with area } F\ (say)\text{ and}$$
$$16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall\ x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy(\sum_{cyc}xy+z^2)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(1)$$

$$\text{We have}:\frac{a^2b^2.\sqrt{ab}(\sqrt{b+c}+\sqrt{c+a})}{b^2c^2.\sqrt{bc}+c^2a^2.\sqrt{ca}}+\frac{b^2c^2.\sqrt{bc}(\sqrt{a+b}+\sqrt{c+a})}{a^2b^2.\sqrt{ab}+c^2a^2.\sqrt{ca}}$$
$$+\frac{c^2a^2.\sqrt{ca}(\sqrt{a+b}+\sqrt{b+c})}{a^2b^2.\sqrt{ab}+b^2c^2.\sqrt{bc}}=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(\begin{matrix}x=a^2b^2.\sqrt{ab},y=b^2c^2.\sqrt{bc},z=c^2a^2.\sqrt{ca},\\A=\sqrt{a+b},B=\sqrt{b+c},C=\sqrt{c+a}\end{matrix}\right)$$
$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$
$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$= \sqrt{3\sum_{cyc}\sqrt{(a+b)(b+c)}} \overset{A-G}{\geq} \sqrt{9.\sqrt[3]{(a+b)(b+c)(c+a)}}$$

$$\overset{Cesaro}{\geq} 3.\sqrt{\sqrt[3]{8abc}} \overset{abc=1}{=} 3\sqrt{2}\ \forall a,b,c>0 \,|\, abc=1, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1494. If $a,b,c>0$ and $abc=1$, then prove that:

$$\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}+\frac{\sqrt{(a+c)(a^3+b^3)}+\sqrt{(a+b)(a^3+c^3)}}{\sqrt{b+c}\left(\sqrt{a+b}+\sqrt{a+c}\right)}$$

$$+\frac{\sqrt{(b+c)(a^3+b^3)}+\sqrt{(a+b)(b^3+c^3)}}{\sqrt{a+c}\left(\sqrt{a+b}+\sqrt{b+c}\right)}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-India

$$\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$=\frac{\sqrt{(c+a)(b+c)(b^2-bc+c^2)}+\sqrt{(b+c)(c+a)(c^2-ca+a^2)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$\geq\frac{\sqrt{(b+c)(c+a)}\left(\frac{b+c}{2}+\frac{c+a}{2}\right)}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}=\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}^2+\sqrt{c+a}^2\right)}{2.\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$\geq\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)^2}{4.\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{c+a}\right)}$$

$$\Rightarrow\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}\geq\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{4.\sqrt{a+b}}$$

$$and\ analogs\Rightarrow LHS\geq\frac{1}{4}\sum_{cyc}\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{\sqrt{a+b}}\overset{A-G}{\geq}$$

$$\frac{3}{4}.\sqrt[3]{\frac{\left(\prod_{cyc}\sqrt{(b+c)(c+a)}\right)\left(\prod_{cyc}\left(\sqrt{b+c}+\sqrt{c+a}\right)\right)}{\prod_{cyc}\sqrt{a+b}}}$$

$$\overset{Cesaro}{\geq}\frac{3}{4}.\sqrt[3]{\sqrt{(a+b)(b+c)(c+a)}.8.\sqrt{(a+b)(b+c)(c+a)}}$$

$$=\frac{3}{2}.\sqrt[3]{(a+b)(b+c)(c+a)}\overset{Cesaro}{\geq}\frac{3}{2}.\sqrt[3]{8abc}\overset{abc=1}{=}3\ \forall a,b,c>0\,|\,abc=1,$$

$$''=''\ \text{iff } a=b=c=1\ (QED)$$

1495. If $a,b,c>0, then:$

$$\sum_{cyc}(a^2-4\ln a)\geq\frac{3}{2}\left(\frac{a+b+c}{3}\right)^2-8.\sqrt{\frac{a+b+c}{3}}+\frac{65}{8}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

Let $f(x)=x^2-4\ln x\ \forall x>0\ \therefore f''(x)=\frac{4}{x^2}+2\Rightarrow f(x)$ is convex

$$\therefore\sum_{cyc}(a^2-4\ln a)\overset{Jensen}{\geq}3\left(\left(\frac{\sum_{cyc}a}{3}\right)^2-4\ln\left(\frac{\sum_{cyc}a}{3}\right)\right)\overset{?}{>}$$

$$\frac{3}{2}\left(\frac{a+b+c}{3}\right)^2-8.\sqrt{\frac{a+b+c}{3}}+\frac{65}{8}$$

$$\Leftrightarrow \frac{3}{2}\left(\frac{\sum_{cyc} a}{3}\right)^2 + 8.\sqrt{\frac{\sum_{cyc} a}{3}} - \frac{65}{8} \overset{?}{>} 12\ln\left(\frac{\sum_{cyc} a}{3}\right)$$

$$\Leftrightarrow \frac{3}{2}.t^4 + 8t - \frac{65}{8} \overset{?}{>} 12t^2 \left(t = \sqrt{\frac{\sum_{cyc} a}{3}}\right) \Leftrightarrow 12t^4 + 64t - 65 \boxed{\underbrace{\overset{?}{>}}_{(*)}} 192\ln t$$

$$\text{Lem } F(m) = \ln m - \left(m - 1 - \frac{(m-1)^2}{2} + \frac{(m-1)^3}{3}\right) \forall\, m > 0$$

$\therefore F'(m) = \frac{(1-m)^3}{m}$. For $m \geq 1, F'(m) \leq 0 \Rightarrow F(m)$ is $\downarrow$ on $[1,\infty) \Rightarrow F(m) \leq F(1) = 0$ *and* for $0 < m \leq 1, F'(m) \geq 0 \Rightarrow F(m)$ is $\uparrow$ on $(0,1] \Rightarrow F(m) \leq F(1) = 0$

$$\therefore F(m) \leq 0\ \forall\, m > 0 \Rightarrow \ln m \leq m - 1 - \frac{(m-1)^2}{2} + \frac{(m-1)^3}{3}\ \forall\, m > 0 \to (1)$$

$\boxed{Case\ 1}$ $0 < t < \frac{3}{5}$ *and then* : $192\ln t < 192\ln\frac{3}{5} \approx -98.0785 < -98$ $\overset{?}{<} 12t^4 + 64t - 65 \Leftrightarrow 12t^4 + 64t + 33 \overset{?}{>} 0 \to$ true $\Rightarrow (*)$ is true

$\boxed{Case\ 2}$ $t \geq \frac{3}{5}$ *and* $192\ln t \leq 192(t-1) - 96(t-1)^2 + 64(t-1)^3 \overset{?}{<}$ $12t^4 + 64t - 65 \Leftrightarrow 12t^4 - 64t^3 + 288t^2 - 512t + 287 \overset{?}{>} 0$

$$\Leftrightarrow 48t^4 - 256t^3 + 1152t^2 - 2048t + 1148 \overset{?}{>} 0$$

$$\Leftrightarrow \frac{1}{16}(48t^2 - 136t + 737)(4t-5)^2 + 7\left(t - \frac{57}{112}\right) \underbrace{\overset{?}{>}}_{(\bullet)} 0$$

Now, discriminant of : $48t^2 - 136t + 737 = 136^2 - 192 * 737 = -123008 < 0$ $\Rightarrow 48t^2 - 136t + 737 > 0$ *and* $\because t \geq \frac{3}{5} \therefore t - \frac{57}{112} > 0 \therefore$ LHS of $(\bullet) > 0$ $\Rightarrow (\bullet) \Rightarrow (*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall\, t > 0$

$$\therefore \sum_{cyc}(a^2 - 4\ln a) > \frac{3}{2}\left(\frac{a+b+c}{3}\right)^2 - 8.\sqrt{\frac{a+b+c}{3}} + \frac{65}{8}\ \text{(QED)}$$

1496. If $a, b, c > 0$ and $\lambda \geq \frac{1}{2}$, then :

$$\frac{a}{\sqrt{ab + \lambda b^2}} + \frac{b}{\sqrt{bc + \lambda c^2}} + \frac{c}{\sqrt{ca + \lambda a^2}} \geq \frac{3}{\sqrt{\lambda + 1}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ *and* $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of *a* triangle with semiperimeter, circumradius *and* inradius $= s, R, r$ (*say*) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y,$

$$c = s - z \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (2)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3)$$

$$\text{Now, } \frac{a}{\sqrt{ab+\lambda b^2}} + \frac{b}{\sqrt{bc+\lambda c^2}} + \frac{c}{\sqrt{ca+\lambda a^2}} = \sum_{cyc} \frac{a^2}{\sqrt{ab}.\sqrt{a^2+\lambda ab}} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\left(\sqrt{ab}.\sqrt{a^2+\lambda ab}\right)} \overset{\text{CBS}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sqrt{\sum_{cyc} ab}.\sqrt{\sum_{cyc} a^2 + \lambda\sum_{cyc} ab}} \overset{?}{\geq} \frac{3}{\sqrt{\lambda+1}}$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} \frac{s^4}{(4Rr+r^2)(s^2-8Rr-2r^2+\lambda(4Rr+r^2))} \overset{?}{\geq} \frac{9}{\lambda+1}$$

$$\Leftrightarrow \lambda\left(s^4 - 9(4Rr+r^2)^2\right) + s^4 - (36Rr+9r^2)(s^2-8Rr-2r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{We have}: s^4 - 9(4Rr+r^2)^2 = (s^2-12Rr-3r^2)(s^2+12Rr+3r^2)$$

$$\overset{\text{Gerretsen + Euler}}{\geq} 0 \text{ and } \because \lambda \geq \frac{1}{2} \therefore \text{LHS of } (*) \geq$$

$$\frac{1}{2}\left(s^4 - 9(4Rr+r^2)^2\right) + s^4 - (36Rr+9r^2)(s^2-8Rr-2r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^4 - (24Rr+6r^2) + 9r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow (s^2-12Rr-3r^2)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{a}{\sqrt{ab+\lambda b^2}} + \frac{b}{\sqrt{bc+\lambda c^2}} + \frac{c}{\sqrt{ca+\lambda a^2}} \geq \frac{3}{\sqrt{\lambda+1}}$$

$$\forall a,b,c > 0 \text{ and } \lambda \geq \frac{1}{2}, \text{"="} \text{ iff } a = b = c \text{ (QED)}$$

1497. If $a,b,c > 0$ and $abc = 1$, then prove that :

$$\frac{a}{\sqrt{b^2+2c}} + \frac{b}{\sqrt{c^2+2a}} + \frac{c}{\sqrt{a^2+2b}} \geq \sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{\sqrt{b^2+2c}} + \frac{b}{\sqrt{c^2+2a}} + \frac{c}{\sqrt{a^2+2b}}$$

$$= \frac{a\sqrt{a}}{\sqrt{b^2a+2ca}} + \frac{b\sqrt{b}}{\sqrt{c^2b+2ab}} + \frac{c\sqrt{c}}{\sqrt{a^2c+2bc}} \overset{\text{Radon}}{\geq} \frac{\left(\sum_{cyc} a\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc} ab^2 + 2\sum_{cyc} ab}} \overset{?}{\geq} \sqrt{3}$$

$$\Leftrightarrow\left(\sum_{cyc}a\right)^3\overset{?}{\geq}3\sum_{cyc}ab^2+6\sum_{cyc}ab$$

$$\Leftrightarrow\sum_{cyc}a^3+6abc+3\sum_{cyc}a^2b+3\sum_{cyc}ab^2\overset{?}{\geq}3\sum_{cyc}ab^2+6\sum_{cyc}ab$$

$$\Leftrightarrow\sum_{cyc}a^3+6abc+3\sum_{cyc}a^2b\ \boxed{\overset{?}{\underset{(\bullet)}{\geq}}}\ 6\sum_{cyc}ab$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^3=\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)$$

$$\overset{via\ (1)}{=}s^3-3xyz\Rightarrow\sum_{cyc}a^3=s^3-12Rrs\to(4)$$

Now, via Bergstrom and via (2), (3) and (4), LHS of $(\bullet)$ − RHS of $(\bullet)$

$$\geq s^3-12Rrs+6r^2s-6(4Rr+r^2)+\frac{3\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}a}\overset{via\ (1)\ and\ (3)}{=}$$

$$\frac{s^2(s^2-12Rr+6r^2)-6s(4Rr+r^2)+3(4Rr+r^2)^2}{s}\overset{?}{\geq}0$$

$$\Leftrightarrow s^2(s^2-12Rr+6r^2)+3(4Rr+r^2)^2\overset{?}{\geq}6s(4Rr+r^2)$$

$$\overset{abc=1}{=}6s(4Rr+r^2).\sqrt[3]{abc}\overset{via\ (2)}{=}6s(4Rr+r^2).\sqrt[3]{r^2s}$$

$$\Leftrightarrow\left(s^2(s^2-12Rr+6r^2)+3(4Rr+r^2)^2\right)^3\overset{?}{\geq}216r^2s^4(4Rr+r^2)^3$$

$$\Leftrightarrow\left(s^2(s^2-12Rr+6r^2)+3(4Rr+r^2)^2\right)^3-216r^2s^4(4Rr+r^2)^3\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0\text{ and}$$

$$\because\text{ via Gerretsen, }P=(s^2-16Rr+5r^2)^6+(60Rr-12r^2)(s^2-16Rr+5r^2)^5$$
$$+2r^2(768R^2-210Rr+21r^2)(s^2-16Rr+5r^2)^4$$
$$+4r^3(5360R^3-1272R^2r+60Rr^2-4r^3)(s^2-16Rr+5r^2)^3$$
$$+4r^4(43008R^4-7536R^2r-4284R^2r^2-228Rr^3-75r^4)(s^2-16Rr+5r^2)^2$$
$$+16r^5\begin{pmatrix}47040R^5-20256R^4r-15636R^3r^2+1032R^2r^3\\+1263Rr^4+132r^5\end{pmatrix}(s^2-16Rr+5r^2)\geq0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\geq P\Leftrightarrow$

$$87808t^6-174144t^5-23952t^4+39428t^3+6198t^2-1875t-338\geq0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{matrix}87808t^4+177088t^3+333168t^2\\+663748t+1328518\end{matrix}\right)+2657205\right)\geq 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \Rightarrow (\bullet) \text{ is true} \therefore \frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}}\geq\sqrt{3}$$

$$\forall\, a,b,c>0 \,|\, abc=1,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1498. If $a+b+c=3$ and $a,b,c\in\mathbb{R}$, then :

$$\frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}\geq 0 \overset{a+b+c=3}{\Leftrightarrow}$$

$$\frac{a^2-bc}{a^2+\frac{(\sum_{cyc}a)^2}{3}}+\frac{b^2-ca}{b^2+\frac{(\sum_{cyc}a)^2}{3}}+\frac{c^2-ab}{c^2+\frac{(\sum_{cyc}a)^2}{3}}\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left((a^2-bc)\left(3b^2+\left(\sum_{cyc}a\right)^2\right)\left(3c^2+\left(\sum_{cyc}a\right)^2\right)\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}a^6+27a^2b^2c^2+3\sum_{cyc}a^4b^2+3\sum_{cyc}a^2b^4 \overset{(\bullet)}{\geq} 9abc\sum_{cyc}a^3+\sum_{cyc}a^3b^3$$

$$+3abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\because x^2+y^2+z^2-xy-yz-zx=\frac{1}{2}\sum_{cyc}(x-y)^2\geq 0\ \forall\, x,y,z\in\mathbb{R}$$

$$\therefore 3\sum_{cyc}a^4b^2+3\sum_{cyc}a^2b^4\geq 3abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\ \forall\, a,b,c\in\mathbb{R}\rightarrow \text{(i)}$$

$\boxed{\text{Case 1}}$ Exactly 2 variables ≤ 0 and WLOG we may assume $b,c\leq 0$ and then :

$$a=3-b-c\geq 3 \therefore -ca\geq 0 \text{ and } -ab\geq 0 \therefore \frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}$$

$$\geq \frac{a^2}{a^2+3}-\frac{bc}{a^2+3} \overset{A-G}{\geq} \frac{a^2}{a^2+3}-\frac{(b+c)^2}{4(a^2+3)} \overset{a+b+c=3}{=} \frac{4a^2-(3-a)^2}{4(a^2+3)}$$

$$=\frac{3(a^2+2a-3)}{4(a^2+3)}=\frac{3(a-1)(a+3)}{4(a^2+3)}>0\ (\because a\geq 3)$$

$$\therefore \frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}>0$$

Case 2 Exactly 1 variable ≤ 0 and WLOG we may assume $a\leq 0$ $(b,c\geq 0)$

and then : $9abc\sum_{cyc}a^3-27a^2b^2c^2$

$$=9abc\left(3abc+\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\right)-27a^2b^2c^2 \overset{a+b+c=3}{=}$$

$$\frac{27}{2}abc\sum_{cyc}(a-b)^2\leq 0\ (\because a\leq 0 \text{ and } b,c\geq 0)\Rightarrow 27a^2b^2c^2\geq 9abc\sum_{cyc}a^3\to \text{(ii)}$$

and via A − G, $\sum_{cyc}a^6>\sum_{cyc}a^3b^3\to$ (iii) (strict inequality for $a\leq 0$ and $b,c\geq 0$)

$$\therefore \text{(i)}+\text{(ii)}+\text{(iii)}\Rightarrow (\bullet) \text{ is true } \therefore \frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}>0$$

Case 3 $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow \sum_{cyc}a=s\to(1)\Rightarrow a=s-x$, $b=s-y, c=s-z \therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow \sum_{cyc}ab=$

$$\sum_{cyc}(s-x)(s-y)\Rightarrow \sum_{cyc}ab=4Rr+r^2\to(3),$$

$$\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab \overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc}a^2=s^2-8Rr-2r^2\to(4),$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow \sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)\to(5)$$

$$\text{Now}, (\bullet)\Leftrightarrow 3a^2b^2c^2+\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^2-3\sum_{cyc}a^2b^2\right)+27a^2b^2c^2$$

$$+3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 9a^2b^2c^2$$

$$\geq 9abc\left(3abc + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)\right) + 3a^2b^2c^2$$

$$+\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9a^2b^2c^2$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^3 \geq 9abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)$$

$$+\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)$$

$$\overset{\text{via (1),(2),(3),(4) and (5)}}{\Leftrightarrow} (s^2 - 8Rr - 2r^2)^3 \geq 9r^2s^2(s^2 - 8Rr - 2r^2 - 4Rr - r^2)$$

$$+(4Rr + r^2)(r^2((4R + r)^2 - 2s^2) - r^2s^2) + 3r^2s^2(4Rr + r^2)$$

$$\Leftrightarrow s^6 - (24Rr + 15r^2)s^4 + r^2s^2(192R^2 + 204Rr + 39r^2) - 9r^3(4R + r)^3 \boxed{\overset{(*)}{\geq}} 0$$

and $\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \geq (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (12R - 15r)s^4 - rs^2(288R^2 - 342Rr + 18r^2)$$

$$+r^2(1760R^3 - 2136R^2r + 546Rr^2 - 67r^3) \boxed{\overset{(**)}{\geq}} 0 \text{ and}$$

$\because (12R - 15r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \geq (12R - 15r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (48R^2 - 129Rr + 66r^2)s^2 \geq r(656R^3 - 1812R^2r + 1077Rr^2 - 154r^3)$$

$$\Leftrightarrow (R - 2r)(48R - 33r)s^2 \geq r(R - 2r)(656R^2 - 500Rr + 77r^2)$$

$$\Leftrightarrow (48R - 33r)s^2 \boxed{\overset{(***)}{\geq}} r(656R^2 - 500Rr + 77r^2) \left(\because R - 2r \overset{\text{Euler}}{\geq} 0\right)$$

Now, $(48R - 33r)s^2 \overset{\text{Gerretsen}}{\geq} (48R - 33r)(16Rr - 5r^2) \overset{?}{\geq}$

$$r(656R^2 - 500Rr + 77r^2) \Leftrightarrow 28R^2 - 67Rr + 22r^2 \overset{?}{\geq} 0$$

$\Leftrightarrow (R - 2r)(28R - 11r) \overset{?}{\geq} 0 \rightarrow$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \Rightarrow (\bullet)$

is true $\therefore \dfrac{a^2 - bc}{a^2 + 3} + \dfrac{b^2 - ca}{b^2 + 3} + \dfrac{c^2 - ab}{c^2 + 3} \geq 0 \;\forall\; a, b, c \in \mathbb{R} \,\middle|\, a + b + c = 3,$

" = " iff $a = b = c = 1$ (QED)

1499. If $x, y, z > 0$ and $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, then prove that :

$$(xy + yz + zx)\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)^2 \geq 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \sqrt{x} = a, \sqrt{y} = b, \sqrt{z} = c \text{ and then : } x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\Rightarrow \sum_{cyc} a^2 = \frac{\sum_{cyc} a^2b^2}{a^2b^2c^2} \Rightarrow 1 = \frac{a^2b^2c^2 \sum_{cyc} a^2}{\sum_{cyc} a^2b^2} \to (i)$$

Assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0$ and $Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - X, b = s - Y, c = s - Z$

$$\therefore abc = r^2s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - X)(s - Y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\text{Now, } (xy + yz + zx)(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})^2 \ge 27 \Leftrightarrow \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} ab\right)^2 \ge 27$$

$$\overset{\text{via (i)}}{=} 27\left(\frac{a^2b^2c^2 \sum_{cyc} a^2}{\sum_{cyc} a^2b^2}\right)^2 \Leftrightarrow \left(\sum_{cyc} a^2b^2\right)^3 \left(\sum_{cyc} ab\right)^2 \ge 27(abc)^4 \left(\sum_{cyc} a^2\right)^2$$

$$\overset{\text{via (2),(3),(4) and (5)}}{\Leftrightarrow} r^6((4R + r)^2 - 2s^2)^3 . r^2(4R + r)^2 \ge 27r^8s^4(s^2 - 8Rr - 2r^2)^2$$

$$\Leftrightarrow ((4R + r)^2 - 2s^2)^3 (4R + r)^2 \boxed{\overset{(*)}{\ge}} 27s^4(s^2 - 8Rr - 2r^2)^2$$

$$\text{Now, LHS of } (*) - \text{RHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$\left((4R + r)^2 - 2(4R^2 + 4Rr + 3r^2)\right)^3 (4R + r)^2$$

$$-27(4R^2 + 4Rr + 3r^2)^2(4R^2 + 4Rr + 3r^2 - 8Rr - 2r^2)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow 160t^8 + 512t^7 - 1856t^6 - 96t^5 + 756t^4 + 600t^3 - 391t^2 + 37t - 46 \overset{?}{\ge} 0$$

$$\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}160t^6+1152t^5+2112t^4+3744t^3\\ +7284t^2+14760t+29513\end{pmatrix}+59049\right)\overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore (xy+yz+zx)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)^2 \geq 27$$

$$\forall x,y,z>0 \,\middle|\, x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\text{''}=\text{''} \text{ iff } x=y=z=1 \text{ (QED)}$$

1500. If $a,b,c>0$ and $ab+bc+ca=1$, then prove that:

$$\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2}\geq\frac{5}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \to (1) \Rightarrow a=s-x, b=s-y, c=s-z$

$$\therefore abc=r^2s \to (2) \text{ and such substitutions } \Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab=4Rr+r^2 \to (3), \sum_{cyc} a^2=\left(\sum_{cyc} a\right)^2-2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2=s^2-8Rr-2r^2 \to (4) \text{ and we have:}$$

$$\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2} \overset{ab+bc+ca=1}{=} \sum_{cyc}\frac{(ab+bc+ca)^2+a^2b^2}{(a+b)^2}$$

$$=\sum_{cyc}\frac{2a^2b^2+c^2(a+b)^2+2abc(a+b)}{(a+b)^2}$$

$$=2\left(\sum_{cyc}\frac{ab}{a+b}\right)^2-4abc\sum_{cyc}\frac{a}{(a+b)(c+a)}+\sum_{cyc} a^2+2abc\sum_{cyc}\frac{1}{a+b}$$

$$=2\left(\frac{1}{\prod_{cyc}(a+b)}\cdot\sum_{cyc}(ab(b+c)(c+a))\right)^2-\frac{4abc}{\prod_{cyc}(a+b)}\cdot\sum_{cyc} a(b+c)+\sum_{cyc} a^2$$

$$+\frac{2abc}{\prod_{cyc}(a+b)}\sum_{cyc}(b+c)(c+a)$$

$$=\frac{2}{\prod_{cyc}(a+b)^2}\left(\left(\sum_{cyc}ab\right)^2+abc\left(\sum_{cyc}a\right)\right)^2-\frac{8abc}{\prod_{cyc}(a+b)}\left(\sum_{cyc}ab\right)+\sum_{cyc}a^2$$

$$+\frac{2abc}{\prod_{cyc}(a+b)}\cdot\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)\overset{\text{via (1),(2),(3) and (4)}}{=}$$

$$\frac{2r^4\left((4R+r)^2+s^2\right)^2}{16R^2r^2s^2}+s^2-8Rr-2r^2-\frac{8r^2sr(4R+r)}{4Rrs}+\frac{2r^2s}{4Rrs}\left(s^2+4Rr+r^2\right)$$

$$=\frac{r^2\left((4R+r)^2+s^2\right)^2}{8R^2s^2}+s^2-8Rr-2r^2+r.\frac{s^2+4Rr+r^2-16Rr-4r^2}{2R}$$

$$=\frac{8R^2s^2\left(s^2-8Rr-2r^2\right)+r^2\left((4R+r)^2+s^2\right)^2+4Rrs^2\left(s^2-12Rr-3r^2\right)}{8R^2s^2}\overset{?}{\geq}\frac{5}{2}$$

$$\overset{ab+bc+ca=1}{=}\frac{5}{2}\left(\sum_{cyc}ab\right)\overset{\text{via (3)}}{=}\frac{5\left(4Rr+r^2\right)}{2}$$

$$\Leftrightarrow\left(8R^2+4Rr+r^2\right)s^4-rs^2\left(144R^3+52R^2r-4Rr^2-2r^3\right)+r^2(4R+r)^4\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0$$

$and \because \left(8R^2+4Rr+r^2\right)\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen}}{\geq}0 \therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\geq\left(8R^2+4Rr+r^2\right)\left(s^2-16Rr+5r^2\right)^2$

$$\Leftrightarrow\left(28R^3-R^2r-Rr^2-2r^3\right)s^2\ \boxed{\overset{(**)}{\geq}}\ r\left(448R^4-128R^3r-70R^2r^2-19Rr^3+6r^4\right)$$

Now, $\left(28R^3-R^2r-Rr^2-2r^3\right)s^2\overset{\text{Rouche}}{\geq}$

$\left(28R^3-R^2r-Rr^2-2r^3\right)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$

$\overset{?}{\geq}r\left(448R^4-128R^3r-70R^2r^2-19Rr^3+6r^4\right)$

$\Leftrightarrow(R-2r)\left(56R^4-58R^3r-28R^2r^2+Rr^3+2r^4\right)\ \boxed{\underset{(***)}{\overset{?}{\geq}}}$

$2(R-2r)\left(28R^3-R^2r-Rr^2-2r^3\right).\sqrt{R^2-2Rr}$

$$\because 56R^4-58R^3r-28R^2r^2+Rr^3+2r^4$$

$$=(R-2r)\left(56R^3+54R^2r+80Rr^2+161r^3\right)+324r^4\overset{\text{Euler}}{\geq}324r^4>0\ and$$

$$\because R-2r\overset{\text{Euler}}{\geq}0 \therefore \text{in order to prove } (***), \text{it suffices to prove :}$$

$$\left(56R^4-58R^3r-28R^2r^2+Rr^3+2r^4\right)^2>4\left(R^2-2Rr\right)\left(28R^3-R^2r-Rr^2-2r^3\right)^2$$

$$\Leftrightarrow 3360t^5-8t^4-264t^3-95t^2+36t+4>0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(3360t^4+6712t^3+13160t^2+26225t+52486\right)+104976>0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{1+a^2b^2}{(a+b)^2} + \frac{1+b^2c^2}{(b+c)^2} + \frac{1+c^2a^2}{(c+a)^2}$$
$$\geq \frac{5}{2}\ \forall a,b,c > 0 \,\middle|\, ab+bc+ca = 1,'' ='' \text{ iff } a = b = c = \frac{1}{\sqrt{3}} \text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru