A NEW PROOF FOR IONESCU - WEITZENBOCK'S INEQUALITY USING THE TORICELLI'S POINT

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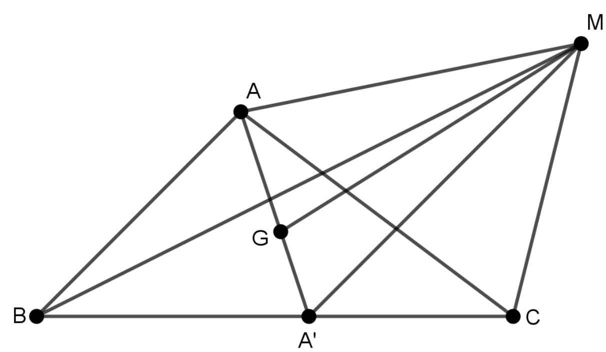
Abstract. In this paper we will find the distance between the Toricelli's point and the centroid in any triangle and we will use this to prove the Ionescu - Weitzenbock's inequality

Lemma.
Let M be an arbitrary point in the plane of ΔABC and G the centroid of ΔABC. In these conditions:

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2$$

Proof.



$$A'B = A'C; AA' - \text{ median in } \Delta ABC$$
$$AB = c; BC = a; CA = b; AA' = m_a$$
$$GA = \frac{2}{3}m_a; GA' = \frac{1}{3}m_a; G \text{ - centroid}$$
$$A'B = A'C = \frac{a}{2}$$
$$MA' - \text{ median in } \Delta MBC:$$

(1) $$MA'^2 = \frac{1}{2}(MB^2 + MC^2) - \frac{1}{4}BC^2$$

We will use Stewart's theorem in $\Delta MAA'$:

$$MG^2 \cdot AA' = MA^2 \cdot GA' + MA'^2 \cdot GA - GA \cdot GA' \cdot AA'$$
$$MG^2 \cdot m_a = MA^2 \cdot \frac{1}{3}m_a + MA'^2 \cdot \frac{2}{3}m_a - \frac{2}{3}m_a \cdot \frac{1}{3}m_a \cdot m_a$$
$$MG^2 = \frac{1}{3}MA^2 + \frac{2}{3}MA'^2 - \frac{2}{9}m_a^2$$

By (1):

$$MG^2 = \frac{1}{3}MA^2 + \frac{2}{3}\left(\frac{1}{2}(MB^2 + MC^2) - \frac{1}{4}BC^2\right) - \frac{2}{9}m_a^2$$

$$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{2}BC^2 - \frac{2}{3}\left(\frac{b^2+c^2}{2} - \frac{a^2}{4}\right)$$

$$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{a^2}{2} - \frac{b^2+c^2}{3} + \frac{a^2}{6}$$

(2) $$MA^2 + MB^2 + MC^2 = 3MG^2 + \frac{1}{3}(a^2 + b^2 + c^2)$$

$$GA^2 + GB^2 + GC^2 = \frac{4}{9}(m_a^2 + m_b^2 + m_c^2) =$$

$$= \frac{4}{9} \cdot \frac{3}{4}(a^2 + b^2 + c^2) = \frac{1}{3}(a^2 + b^2 + c^2)$$

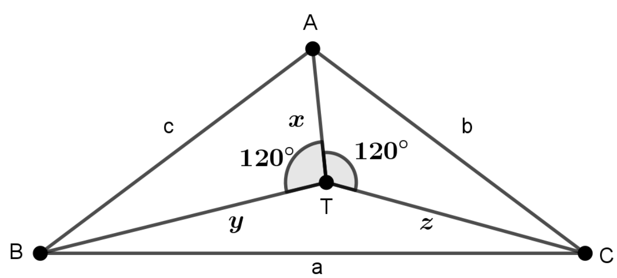
Replacing in (2):

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2$$

□

Back to the main problem:
We take $M = T$ - Toricelli's point in lemma.

(3) $$TA^2 + TB^2 + TC^2 = 3TG^2 + GA^2 + GB^2 + GC^2$$



Denote: $TA = x; TB = y; TC = z$
Replacing in (3):

$$x^2 + y^2 + z^2 = 3TG^2 + GA^2 + GB^2 + GC^2$$

$$3TG^2 = x^2 + y^2 + z^2 - (GA^2 + GB^2 + GC^2)$$

$$TG^2 = \frac{1}{3}(x^2 + y^2 + z^2) - \frac{1}{3} \cdot \frac{4}{9}(m_a^2 + m_b^2 + m_c^2)$$

$$TG^2 = \frac{1}{3}(x^2 + y^2 + z^2) - \frac{4}{27} \cdot \frac{3}{4}(a^2 + b^2 + c^2)$$

(4) $$TG^2 = \frac{1}{3}(x^2 + y^2 + z^2) - \frac{1}{9}(a^2 + b^2 + c^2)$$

$$[ATB] = \frac{1}{2}TA \cdot TB \cdot \sin(\angle ATB)$$

$$[ATB] = \frac{1}{2}xy\sin 120^\circ = \frac{xy}{2}\sin(180^\circ - 60^\circ)$$

$$[ATB] = \frac{xy}{2} \cdot \sin 60^\circ = \frac{xy}{2} \cdot \frac{\sqrt{3}}{2} = \frac{xy\sqrt{3}}{4}$$

$$[ATB] = \frac{xy\sqrt{3}}{4} \tag{5}$$

Analogous:

$$[BTC] = \frac{yz\sqrt{3}}{4} \tag{6}$$

$$[CTA] = \frac{zx\sqrt{3}}{4} \tag{7}$$

By adding $(5); (6); (7)$:

$$[ATB] + [BTC] + [CTA] = \frac{\sqrt{3}}{4}(xy + yz + zx)$$

$$[ABC] = \frac{\sqrt{3}}{4}(xy + yz + zx)$$

$$F = \frac{\sqrt{3}}{4}(xy + yz + zx)$$

$$xy + yz + zx = \frac{4F}{\sqrt{3}} \tag{8}$$

By cosine's law in ΔATB:

$$BC^2 = TA^2 + TB^2 - 2TATB\cos(\angle ATB)$$

$$a^2 = x^2 + y^2 - 2xy\cos 120^\circ$$

$$a^2 = x^2 + y^2 - 2xy\cos(180^\circ - 60^\circ)$$

$$a^2 = x^2 + y^2 - 2xy(-\cos 60^\circ)$$

$$a^2 = x^2 + y^2 + xy \tag{9}$$

Analogous:

$$b^2 = y^2 + z^2 + yz \tag{10}$$

$$c^2 = z^2 + x^2 + zx \tag{11}$$

By adding $(9); (10); (11)$:

$$a^2 + b^2 + c^2 = 2(x^2 + y^2 + z^2) + (xy + yz + zx)$$

By (8):

$$a^2 + b^2 + c^2 = 2(x^2 + y^2 + z^2) + \frac{4F}{\sqrt{3}}$$

$$2(x^2 + y^2 + z^2) = a^2 + b^2 + c^2 - \frac{4F}{\sqrt{3}}$$

$$x^2 + y^2 + z^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2F}{\sqrt{3}} \tag{12}$$

Replacing (12) in (4):

$$TG^2 = \frac{1}{3}\Big(\frac{1}{2}(a^2 + b^2 + c^2) - \frac{2F}{\sqrt{3}}\Big) - \frac{1}{9}(a^2 + b^2 + c^2)$$

$$TG^2 = \frac{1}{6}(a^2 + b^2 + c^2) - \frac{2F}{3\sqrt{3}} - \frac{1}{9}(a^2 + b^2 + c^2)$$

$$TG^2 = \frac{3(a^2+b^2+c^2) - 2(a^2+b^2+c^2)}{18} - \frac{2F\sqrt{3}}{9}$$
$$TG^2 = \frac{a^2+b^2+c^2-4F\sqrt{3}}{18}$$
$$TG^2 \geq 0 \Rightarrow \frac{a^2+b^2+c^2-4F\sqrt{3}}{18} \geq 0$$
$$\Rightarrow a^2+b^2+c^2-4F\sqrt{3} \geq 0$$
$$a^2+b^2+c^2 \geq 4F\sqrt{3}$$

which is Ionescu - Weitzenbock's inequality.

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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