A SIMPLE PROOF FOR SANDOR'S INEQUALITY

DANIEL SITARU, HIKMAT MAMMADOV

Abstract. In this paper we will give a simple proof for Sandor's inequality and a few applications.

SANDOR'S INEQUALITY

If $x > 0$ then:

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x} \tag{1}$$

Proof.

$$\text{Let be } f : [0, \infty) \to \mathbb{R}; f(x) = \sinh x - x$$
$$f'(x) = \cosh x - 1$$
$$f'(x) = 0 \Rightarrow \cosh x - 1 = 0 \Rightarrow \cosh x = 1$$
$$\frac{e^x - e^{-x}}{2} - 1 = 0 \Rightarrow e^x - 2 + e^{-x} = 0$$
$$(\sqrt{e^x} - \sqrt{e^{-x}})^2 = 0 \Rightarrow \sqrt{e^x} = \sqrt{e^{-x}}$$
$$e^x = e^{-x} \Rightarrow x = -x \Rightarrow 2x = 0 \Rightarrow x = 0$$
$$\min_{x \geq 0} f(x) = f(0) \Rightarrow f(x) \geq 0; (\forall) x \geq 0$$
$$\sinh x - x \geq 0 \Rightarrow \sinh x \geq x$$
$$\sinh^2 x \geq x^2 \Rightarrow 1 + \sinh^2 x \geq 1 + x^2$$
$$\cosh^2 x \geq 1 + x^2 \Rightarrow \sqrt[4]{\cosh^2 x} \geq \sqrt[4]{1 + x^2}$$
$$\sqrt{\cosh x} \geq \sqrt[4]{1 + x^2}$$

$$\frac{\sqrt{\cosh x}}{\sqrt[4]{1 + x^2}} \geq 1 \tag{2}$$

By Cauchy - Schwarz's inequality (integral form):

$$\int_0^x (\sqrt{\cosh x})^2 dx \cdot \int_0^x \left(\frac{1}{\sqrt[4]{1 + x^2}}\right)^2 dx \geq$$
$$\geq \left(\int_0^x \frac{\sqrt{\cosh x}}{\sqrt[4]{1 + x^2}} dx\right)^2 \overset{(2)}{\geq} \left(\int_0^x dx\right)^2 = x^2$$
$$\int_0^x \cosh x dx \cdot \int_0^x \frac{1}{\sqrt{1 + x^2}} dx \geq x^2$$
$$\sinh x \cdot \ln(x + \sqrt{1 + x^2}) \geq x^2$$
$$\sinh x \cdot \operatorname{arcsinh} x \geq x^2$$

For $x > 0$ we obtain (1):

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x}$$

□

Application 1.
If $0 < a \le b$ then:

$$e^a + e^{-a} + 2\int_a^b \frac{x^2}{\ln(x+\sqrt{1+x^2})}dx \le e^b + e^{-b}$$

Solution.
By (1):

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x} \Rightarrow \frac{x^2}{\operatorname{arcsinh} x} < \sinh x$$

$$\int_a^b \frac{x^2}{\operatorname{arcsinh} x}dx \le \int_a^b \sinh x dx$$

$$\int_a^b \frac{x^2}{\ln(x+\sqrt{1+x^2})}dx \le \cosh b - \cosh a$$

$$\cosh a + \int_a^b \frac{x^2}{\ln(x+\sqrt{1+x^2})}dx \le \cosh b$$

$$\frac{e^a + e^{-a}}{2} + \int_a^b \frac{x^2}{\ln(x+\sqrt{1+x^2})}dx \le \frac{e^b + e^{-b}}{2}$$

$$e^a + e^{-a} + 2\int_a^b \frac{x^2}{\ln(x+\sqrt{1+x^2})}dx \le e^b + e^{-b}$$

□

Application 2.
In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sinh A\right)\left(\sum_{cyc} \operatorname{arcsinh} A\right) > \pi^2$$

Solution.

$$\sinh A + \sinh B + \sinh C \overset{(1)}{>} \frac{A^2}{\operatorname{arcsinh} A} + \frac{B^2}{\operatorname{arcsinh} B} + \frac{C^2}{\operatorname{arcsinh} C} \overset{\text{BERGSTRÖM}}{\ge}$$

$$\ge \frac{(A+B+C)^2}{\operatorname{arcsinh} A + \operatorname{arcsinh} B + \operatorname{arcsinh} C} = \frac{\pi^2}{\operatorname{arcsinh} A + \operatorname{arcsinh} B + \operatorname{arcsinh} C}$$

$$\left(\sum_{cyc} \sinh A\right)\left(\sum_{cyc} \operatorname{arcsinh} A\right) > \pi^2$$

□

Application 3.
In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sinh a\right)\left(\sum_{cyc} \operatorname{arcsinh} a\right) > 108r^2$$

Solution.

$$\sinh a + \sinh b + \sinh c \overset{(1)}{>} \frac{a^2}{\operatorname{arcsinh} a} + \frac{b^2}{\operatorname{arcsinh} b} + \frac{c^2}{\operatorname{arcsinh} c} \geq$$

$$\overset{\text{BERGSTRÖM}}{\geq} \frac{(a+b+c)^2}{\operatorname{arcsinh} a + \operatorname{arcsinh} b + \operatorname{arcsinh} c}$$

$$\left(\sum_{cyc} \sinh a\right)\left(\sum_{cyc} \operatorname{arcsinh} a\right) > (a+b+c)^2 = 4s^2 \geq$$

$$\overset{\text{MITRINOVIC}}{\geq} 4 \cdot (3\sqrt{3})^2 \cdot r^2 = 108r^2$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com