A NEW GENERALIZATION FOR GORDON'S INEQUALITY

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ABSTRACT. In this paper we will give a generalization for Gordon's inequality.

Main result:
If $m \geq 0, M \in Int(\Delta ABC); d_a = d(A, BC); d_b = d(B, AC); d_c = d(C, AB)$ then in ΔABC the following relationship holds:

$$\frac{a^{m+1} \cdot b}{d_b^m} + \frac{b^{m+1} \cdot c}{d_c^m} + \frac{c^{m+1} \cdot a}{d_a^m} \geq 2^{m+2} \cdot (\sqrt{3})^{m+1} \cdot F \tag{1}$$

If $m = 0$ then (1) becomes the classical Goldner's inequality:

$$ab + bc + ca \geq 4\sqrt{3}F$$

Proof 1.

Denote: $F_a = [MBC]; F_b = [MCA]; F_c = [MAB]$

$$\sum_{cyc} \frac{a^{m+1} \cdot b}{d_b^m} = \sum_{cyc} \frac{a^{m+1} \cdot b^{m+1}}{(b \cdot d_b)^m} = \sum_{cyc} \frac{(ab)^{m+1}}{(2F_b)^m} =$$

$$= \frac{1}{2^m} \sum_{cyc} \frac{(ab)^{m+1}}{F_b^m} \overset{\text{RADON}}{\geq} \frac{(ab + bc + ca)^{m+1}}{2^m (F_a + F_b + F_c)^m} =$$

$$= \frac{(s^2 + 4Rr + r^2)^{m+1}}{2^m \cdot F^m} = \frac{1}{(2F)^m} (s^2 + r(4R + r))^{m+1} \geq$$

$$\overset{\text{DOUCET}}{\geq} \frac{1}{(2F)^m} \cdot (s^2 + s\sqrt{3} \cdot r)^{m+1} \geq$$

$$\overset{\text{MITRINOVIC}}{\geq} \frac{1}{(2F)^m} \cdot (s \cdot 3\sqrt{3}r + sr\sqrt{3})^{m+1} =$$

$$= \frac{1}{2^m \cdot F^m} \cdot (3\sqrt{3}F + \sqrt{3}F)^{m+1} =$$

$$= \frac{1}{2^m \cdot F^m} \cdot 4^{m+1} \cdot (\sqrt{3})^{m+1} \cdot F^{m+1} =$$

$$= 2^{m+2} \cdot (\sqrt{3})^{m+1} \cdot F$$

□

Proof 2.

$$\sum_{cyc} \frac{a^{m+1} \cdot b}{d_b^m} = \sum_{cyc} \frac{a^{m+1} \cdot b^{m+1}}{(b \cdot d_b)^m} = \sum_{cyc} \frac{(ab)^{m+1}}{(2F_b)^m} =$$

$$= \frac{1}{2^m} \sum_{cyc} \frac{(ab)^{m+1}}{F_b^m} = \frac{1}{2^m \cdot F^m} \cdot (ab + bc + ca)^{m+1} \geq$$

$$\overset{\text{AM-GM}}{\geq} \frac{1}{2^m \cdot F^m} \cdot (3\sqrt[3]{(abc)^2})^{m+1} \overset{\text{CARLITZ}}{\geq}$$

$$\geq \frac{1}{2^m \cdot F^m} \cdot (4\sqrt{3}F)^{m+1} = \frac{2^{m+2} \cdot 3^{\frac{m+1}{2}} \cdot F^{m+1}}{2^m \cdot F^m} =$$
$$= 2^{m+2} \cdot (\sqrt{3})^{m+1} \cdot F$$

Equality holds for $a = b = c$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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