ABOUT THE PROBLEM 3326 - CRUX MATHEMATICORUM

D.M. BĂTINEŢU - GIURGIU, CLAUDIA NĂNUŢI - ROMANIA

Abstract. In the mathematical magazine CRUX MATHEMATICORUM from Canada, Volume 34, number 3, it is published the problem 3326 - author Mihály Bencze. In this paper we will give a generalization of this problem.

Problem 3326.
If $a, b, c > 0$ then:

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq 6(a+b+c)^2$$

Mihály Bencze

Generalization
If $a, b, c, s, t, x, y > 0$ then:

$$t^2(xa^2+s)(xb^2+s)(xc^2+s)+s^2(xa^2+t)(yb^2+t)(zc^2+t) \geq$$

$$\geq \frac{3}{4}s^2t^2(x+y)(a+b+c)^2 \tag{1}$$

Proof.
Lemma 1: If $u, v > 0$ then:

$$(u^2+1)(v^2+1) \geq \frac{3}{4}((u+v^2)+1) \tag{2}$$

Solution.

$$u^2v^2+u^2+v^2+1 \geq \frac{3}{4}u^2+\frac{3}{4}v^2+\frac{3}{2}uv+\frac{3}{4}$$

$$u^2v^2+\frac{1}{4}u^2+\frac{1}{4}v^2-\frac{3}{2}uv+\frac{1}{4} \geq 0$$

$$4u^2v^2+u^2+v^2-6uv+1 \geq 0$$

$$4u^2v^2-4uv+1+u^2-2uv+v^2 \geq 0$$

$$(2uv-1)^2+(u-v)^2 \geq 0$$

Equality holds for $u = v = \frac{\sqrt{2}}{2}$. □

Lemma 2: If $u, v, w > 0$ then:

$$(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2 \tag{3}$$

Solution.

$$(u^2+1)(v^2+1)(w^2+1) \overset{(2)}{\geq} \frac{3}{4}((u+v^2)+1)(w^2+1) =$$

$$= \frac{3}{4}((u+v)^2+1^2)(1^2+w^2) \overset{\text{CBS}}{\geq}$$

$$\geq \frac{3}{4}((u+v)\cdot 1+1\cdot w)^2 = \frac{3}{4}(u+v+w)^2$$

□

Back to the main proof:

$$t^2\prod_{cyc}(xa^2+s)+s^2\prod_{cyc}(ya^2+t)=$$
$$=t^2s^3\prod_{cyc}\Big(\frac{xa^2}{s}+1\Big)+s^2t^3\prod_{cyc}\Big(\frac{ya^2}{t}+1\Big)=$$
$$=t^2s^3\prod_{cyc}\Big(\Big(\sqrt{\frac{x}{s}}\cdot a\Big)^2+1\Big)+s^2t^3\prod_{cyc}\Big(\Big(\sqrt{\frac{y}{t}}\cdot a\Big)^2+1\Big)\geq$$
$$\overset{(3)}{\geq} t^2s^3\cdot\frac{3}{4}\Big(\sqrt{\frac{x}{s}}(a+b+c)\Big)^2+s^2t^3\cdot\frac{3}{4}\Big(\sqrt{\frac{y}{t}}(a+b+c)\Big)^2$$
$$=\frac{3}{4}s^3t^2\cdot\frac{x}{s}(a+b+c)^2+\frac{3}{4}s^2t^3\cdot\frac{y}{t}(a+b+c)^2=$$
$$=\frac{3}{4}s^2t^2x(a+b+c)^2+\frac{3}{4}s^2t^2y(a+b+c)^2=$$
$$=\frac{3}{4}s^2t^2(x+y)(a+b+c)^2$$

If we take $x=y$ in (1):

$$t^2(xa^2+s)(xb^2+s)(xc^2+s)+s^2(xa^2+t)(xb^2+t)(zc^2+t)\geq$$
$$\geq\frac{3}{2}s^2t^2(x+y)(a+b+c)^2 \tag{4}$$

If we take $x=1$ in (4):

$$t^2(a^2+s)(b^2+s)(c^2+s)+s^2(a^2+t)(b^2+t)(c^2+t)\geq$$
$$\geq\frac{3}{2}s^2t^2(a+b+c)^2 \tag{5}$$

If we take $s=2;t=1$ in (5) we obtain the problem: 3326:

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1)\geq 6(a+b+c)^2$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com