ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$3\sum_{cyc}\frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \cdot \sinh y}} \ge \left(\sum_{cyc}\sqrt{|x-y|}\right)^2$$

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If $g(x)$ is a convex function on D, then $\forall a, b \in D: g(a) \ge g(b) + g'(b)(a-b)$

The hyperbolic $\sinh$ function is convex on $\mathbb{R}_{>0}$: $\sinh(a) \ge \sinh(b) + \cosh(b)(a-b)$.

W.L.O.G consider $x > y > 0$.

$$\cosh x - \cosh y = \int_y^x \sinh t\,dt \ge \int_y^x\left(\sinh\left(\frac{x+y}{2}\right) + \cosh\left(\frac{x+y}{2}\right)\left(t - \frac{x+y}{2}\right)\right)dt =$$

$$= \sinh\left(\frac{x+y}{2}\right)(x-y)$$

$$\sinh^2\left(\frac{x+y}{2}\right) - \underbrace{\sinh x \sinh y}_{\substack{=\frac{1}{2}(\cosh(x+y)-\cosh(x-y)) \\ Also, \cosh\alpha = 2\cosh^2\left(\frac{\alpha}{2}\right)-1}} =$$

$$= \sinh^2\left(\frac{x+y}{2}\right) - \left(\cosh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right)\right)$$

$$= \cosh^2\left(\frac{x-y}{2}\right) - 1 = \sinh^2\left(\frac{x-y}{2}\right) \ge 0 \Rightarrow \sinh\left(\frac{x+y}{2}\right) \ge \sqrt{\sinh x \sinh y}$$

Therefore, $\forall x, y \in \mathbb{R}_{>0}: \frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \sinh y}} \ge |x-y|$

$$3\sum_{cyc}\frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \sinh y}} \ge 3\sum_{cyc}|x-y| \overset{Cauchy-Schwarz}{\ge} \left(\sum_{cyc}\sqrt{|x-y|}\right)^2$$

The inequality is not always true if the power is changed from 2 to $\frac{1}{2}$.