ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a, b, c \leq 1$ then:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9+2(a^2+b^2+c^2-ab-bc-ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Syria

$$e_1 = 3+\left(\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}\right)-2(a^2+b^2+c^2)+2(ab+bc+ca) \overset{?}{\geq} 9$$

$$\left(\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}\right)-2(a^2+b^2+c^2)+2(ab+bc+ca) \overset{?}{\geq} 6$$

$$\left.\begin{array}{r} by\ AM-GM: \frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b} \geq 6 \\ by\ Cauchy\ Schwarz: 2(a^2+b^2+c^2)-2(ab+bc+ca) \geq 0 \end{array}\right\} \Rightarrow$$

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{a}-2(a^2+b^2+c^2)+2(ab+bc+ca) \geq 6$$

Equality occurred when $a=b=c=1$.

Solution 2 by Ravi Prakash-New Delhi-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-9-2(a^2+b^2+c^2-ab-bc-ca)$$

$$=\left(\frac{b}{a}+\frac{a}{b}-2\right)+\left(\frac{c}{a}+\frac{a}{c}-2\right)+\left(\frac{b}{c}+\frac{c}{b}-2\right)-[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$=(a-b)^2\left(\frac{1}{ab}-1\right)+(b-c)^2\left(\frac{1}{bc}-1\right)+(c-a)^2\left(\frac{1}{ca}-1\right)$$

$$=\frac{(a-b)^2(1-ab)}{ab}+\frac{(b-c)^2(1-bc)}{bc}+\frac{(c-a)^2(1-ca)}{ca} \geq 0 \quad [\because 0 < a,b,c \leq 1]$$

Equality when $a=b=c=1$.