

# ROMANIAN MATHEMATICAL MAGAZINE

If  $a, b, c > 0$  such that  $ab, bc, ca > 1$ , then

$$a^3(bc - 1) + b^3(ca - 1) + c^3(ab - 1) \leq \frac{4}{9}a^3b^3c^3$$

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Lemma : for  $x > 0$ , we have  $x - 1 \leq \frac{4}{27}x^3$ .

Proof: By AM – GM inequality, we have

$$\frac{4}{27}x^3 + 1 = \frac{4}{27}x^3 + \frac{1}{2} + \frac{1}{2} \geq 3 \sqrt[3]{\frac{4}{27}x^3 \cdot \frac{1}{2} \cdot \frac{1}{2}} = x,$$

with equality for  $\frac{4}{27}x^3 = \frac{1}{2}$  or  $x = \frac{3}{2}$ . Using this lemma, we have

$$\begin{aligned} & a^3(bc - 1) + b^3(ca - 1) + c^3(ab - 1) \leq \\ & \leq a^3 \cdot \frac{4}{27}(bc)^3 + b^3 \cdot \frac{4}{27}(ca)^3 + c^3 \cdot \frac{4}{27}(ab)^3 = \frac{4}{9}a^3b^3c^3. \end{aligned}$$

Equality holds for  $a = b = c = \sqrt{\frac{3}{2}}$ .