ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $a^3b^3(a^4+b^4) \geq 2$, then prove that :

$$a^2+b^2 \geq 2$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$a^3b^3(a^4+b^4) \geq 2 > 0 \text{ and } \because a^4+b^4 > 0\ (a^4+b^4 \neq 0) \therefore a^3b^3 > 0$$
$$\Rightarrow ab > 0 \rightarrow (1)$$
$$\text{Now}, \forall\, x, y \in \mathbb{R}, (x+y)^4 = (x^2+y^2+2xy)^2 \geq 4(x^2+y^2)(2xy)$$

$$(\because (m+n)^2 \geq 4mn\ \forall\, m, n \in \mathbb{R}) \therefore \text{choosing } x \equiv a^2, y \equiv b^2,$$

$$(a^2+b^2)^4 \geq 4(a^4+b^4)(2a^2b^2) \rightarrow (\text{i}) \text{ and } a^2+b^2 \geq 2ab \rightarrow (\text{ii})$$
$$\text{and} \because ab \overset{\text{via }(1)}{>} 0 \therefore (\text{i}).(\text{ii}) \Rightarrow (a^2+b^2)^5 \geq 16a^3b^3(a^4+b^4) \overset{a^3b^3(a^4+b^4)\geq 2}{\geq} 32$$

$$\therefore a^2+b^2 \geq 2\ \forall\, a, b \in \mathbb{R} \,\middle|\, a^3b^3(a^4+b^4) \geq 2,$$

$$'' ='' \text{ iff } (a=b=1) \text{ or } (a=b=-1) \text{ (QED)}$$