ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} \in [3,5]$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 50$, then prove that :

$$a + \mathrm{b} + \mathrm{c} \geq 12$$

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$a, \mathrm{b}, \mathrm{c} \in [3,5] \Rightarrow 0 \leq a-3, \mathrm{b}-3, \mathrm{c}-3 \leq 2$ and assigning $x = a - 3$, $\mathrm{y} = \mathrm{b} - 3, \mathrm{z} = \mathrm{c} - 3$ with $0 \leq x, \mathrm{y}, \mathrm{z} \leq 2$, we get :

$$\sum_{\mathrm{cyc}} (x+3)^2 = 50 \left(\because \sum_{\mathrm{cyc}} a^2 = 50 \right) \Rightarrow \boxed{\sum_{\mathrm{cyc}} x^2 + 6\sum_{\mathrm{cyc}} x = 23} \to (1) \text{ and } (1) \Rightarrow$$

all of $x, \mathrm{y}, \mathrm{z}$ cannot be zero *and* so, we now focus on the scenario when 2 among $x, \mathrm{y}, \mathrm{z}$ equal zero *and* WLOG we may assume $\mathrm{y} = \mathrm{z} = 0$ *and* then :
$(1) \Rightarrow x^2 + 6x - 23 = 0$ whose non − negative root ≈ 2.65685; but $0 \leq x \leq 2$

$\therefore$ 2 among $x, \mathrm{y}, \mathrm{z}$ cannot be equal to zero and hence, we now focus on the scenario when exactly one among $x, \mathrm{y}, \mathrm{z}$ equals zero and WLOG we may assume $x = 0$ and then : $(1) \Rightarrow \mathrm{y}^2 + \mathrm{z}^2 + 6\mathrm{y} + 6\mathrm{z} = 23 \to (\mathrm{i})$

• If possible, let us assume : $\mathrm{y} < 1$ and then : $6\mathrm{y} + \mathrm{y}^2 < 7\ (\because \mathrm{y} > 0) \overset{\text{via (i)}}{\Rightarrow}$
$23 - \mathrm{z}^2 - 6\mathrm{z} < 7 \Rightarrow \mathrm{z}^2 + 6\mathrm{z} - 16 > 0 \Rightarrow (\mathrm{z}+8)(\mathrm{z}-2) > 0 \to$ impossible
$\because \mathrm{z} - 2 \leq 0$ and $\mathrm{z} + 8 > 0 \Rightarrow (\mathrm{z}+8)(\mathrm{z}-2) \leq 0 \therefore \mathrm{y} \nless 1$

• If possible, let us assume : $\mathrm{y} > 1$ and then : $\mathrm{y}^2 + \mathrm{z}^2 + 6\mathrm{y} + 6\mathrm{z} = 23 \Rightarrow$
$\mathrm{y}^2 - 2\mathrm{y} + 1 + \mathrm{z}^2 - 4\mathrm{z} + 4 + 8\mathrm{y} + 10\mathrm{z} + 18 = 23 \Rightarrow 8\mathrm{y} + 10\mathrm{z}$
$= 5 - (\mathrm{y}-1)^2 - (\mathrm{z}-2)^2 < 5 \Rightarrow 5 > 8\mathrm{y} + 10\mathrm{z} > 8 + 10\mathrm{z} \Rightarrow 10\mathrm{z} < -3 \to$

impossible $\because \mathrm{z} > 0 \therefore \mathrm{y} \ngtr 1 \therefore \mathrm{y} \nless 1, \mathrm{y} \ngtr 1 \Rightarrow \mathrm{y} = 1$ when $x = 0$ and putting $x = 0, \mathrm{y} = 1$ in (i), we get : $\mathrm{z}^2 + 6\mathrm{z} - 16 = 0 \Rightarrow (\mathrm{z}+8)(\mathrm{z}-2) = 0 \Rightarrow \mathrm{z} = 2$ $(\because \mathrm{z} > 0)$ and $x = 0, \mathrm{y} = 1, \mathrm{z} = 2 \Rightarrow x + \mathrm{y} + \mathrm{z} = 3 \Rightarrow a - 3 + \mathrm{b} - 3 + \mathrm{c} - 3 = 3$
$\Rightarrow a + \mathrm{b} + \mathrm{c} = 12 \Rightarrow \boxed{\text{equality case for } a = 3, \mathrm{b} = 4, \mathrm{c} = 5 \text{ and permutations}}$

The final scenario that remains is : $0 < x, \mathrm{y}, \mathrm{z} \leq 2$ and then :

$$(x-2)(\mathrm{y}-2)(\mathrm{z}-2) \leq 0 \Rightarrow x\mathrm{yz} + 4\sum_{\mathrm{cyc}} x - 2\sum_{\mathrm{cyc}} x\mathrm{y} - 8 \leq 0$$

$$\Rightarrow -2\sum_{\mathrm{cyc}} x\mathrm{y} \leq 8 - 4\sum_{\mathrm{cyc}} x - x\mathrm{yz} \overset{x,\mathrm{y},\mathrm{z} > 0 \Rightarrow x\mathrm{yz} > 0}{<} 8 - 4\sum_{\mathrm{cyc}} x \Rightarrow \sum_{\mathrm{cyc}} x^2 + 6\sum_{\mathrm{cyc}} x$$

$$= \left(\sum_{\mathrm{cyc}} x\right)^2 - 2\sum_{\mathrm{cyc}} x\mathrm{y} + 6\sum_{\mathrm{cyc}} x < \left(\sum_{\mathrm{cyc}} x\right)^2 + 8 - 4\sum_{\mathrm{cyc}} x + 6\sum_{\mathrm{cyc}} x \overset{\text{via (1)}}{\Rightarrow}$$

$$23 < t^2 + 2t + 8 \left(t = \sum_{cyc} x\right) \Rightarrow t^2 + 2t - 15 > 0 \Rightarrow (t-3)(t+5) > 0$$

$$\Rightarrow t > 3 \left(\because t = \sum_{cyc} x > 0\right) \therefore \sum_{cyc}(a-3) > 3 \Rightarrow a + b + c > 12$$

$$\therefore a + b + c \geq 12 \ \forall a, b, c \in [3,5] \,\Big|\, a^2 + b^2 + c^2 = 50,$$

$$'' ='' \text{ iff } (a = 3, b = 4, c = 5) \text{ or } (a = 3, b = 5, c = 4) \text{ or } (a = 4, b = 5, c = 3)$$

$$\text{or } (a = 4, b = 3, c = 5) \text{ or } (a = 5, b = 3, c = 4) \text{ or } (a = 5, b = 4, c = 3) \text{ (QED)}$$