ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $(a+1)(b+1) = 4$, then prove that :

$$a^4 + b^4 \ge a^3 + b^3$$

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As $(a+1)(b+1) = 4 \therefore$ either $(a+1),(b+1) > 0$ or
$(a+1),(b+1) < 0$ and in the latter case, $a, b < -1 < 0$
$\Rightarrow a^3 + b^3 < 0 \le a^4 + b^4 \therefore a^4 + b^4 > a^3 + b^3$

When $(a+1),(b+1) > 0$, $(a+1)(b+1) = 4 \Rightarrow \sqrt{(a+1)(b+1)} = 2 \Rightarrow 2 \overset{A-G}{\le} \dfrac{a+1+b+1}{2}$ $(\because (a+1)(b+1) = 4 \Rightarrow (a+1),(b+1) > 0) \Rightarrow a + b \ge 2$

$$\Rightarrow 3 - ab \ge 2 \left(\because (a+1)(b+1) = 4 \Rightarrow a + b + ab \overset{(*)}{=} 3\right) \Rightarrow x = ab \le 1 \to (1)$$

$$\text{Now, } a^4 + b^4 \overset{?}{\ge} a^3 + b^3 \Leftrightarrow (a^2+b^2)^2 - 2a^2b^2 \overset{?}{\ge} (a+b)(a^2+b^2-ab)$$

$$\Leftrightarrow ((a+b)^2 - 2ab)^2 - 2a^2b^2 \overset{?}{\ge} (a+b)((a+b)^2 - 3ab)$$

$$\overset{\text{via }(*)}{\Leftrightarrow} ((3-x)^2 - 2x)^2 - 2x^2 \overset{?}{\ge} (3-x)((3-x)^2 - 2x)$$

$$\Leftrightarrow x^4 - 15x^3 + 68x^2 - 108x + 54 \overset{?}{\ge} 0 \Leftrightarrow (x-1)\left(x^2(x-14) + 54(x-1)\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because (x-1) \overset{\text{via }(1)}{\le} 0 \text{ and } x^2(x-14) \overset{\text{via }(1)}{<} 0 \text{ and } 54(x-1) \overset{\text{via }(1)}{\le} 0$$

$$\Rightarrow \left(x^2(x-14) + 54(x-1)\right) < 0 \therefore a^4 + b^4 \ge a^3 + b^3 \therefore \text{combining all cases,}$$

$$a^4 + b^4 \ge a^3 + b^3 \ \forall\, a, b \in \mathbb{R} \,\middle|\, (a+1)(b+1) = 4, \text{"="} \text{ iff } a = b = 1 \text{ (QED)}$$