ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in [3;5]$ and $a^2 + b^2 + c^2 = 50$, then prove that:

$$a + b + c \geq 12$$

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$$* \ a, b, c \in [3, 5] \Rightarrow (a-3)(b-3)(c-3) \geq 0$$

and $(5-a)(5-b)(5-c) \geq 0$

$$\Rightarrow (a-3)(b-3)(c-3) + (5-a)(5-b)(5-c) \geq 0$$

Expand and simplify:

$$\Leftrightarrow 2(ab + bc + ca) - 16(a + b + c) + 98 \geq 0$$

But: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$$\Leftrightarrow (a+b+c)^2 - (a^2 + b^2 + c^2) - 16(a+b+c) + 98 \geq 0$$

$$\Leftrightarrow (a+b+c)^2 - 16(a+b+c) + 48 \geq 0$$

$$\Leftrightarrow a + b + c \geq 12 \text{ or } a + b + c \leq 4$$

$a, b, c \in [3, 5] \Rightarrow a + b + c \geq 12$, proved.

Equality holds iff: $a = 3, b = 4, c = 5$ and permutations.