ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=3$, then prove that:

$$a^3+b^3+c^3+\sqrt{5}abc\ge 3+\sqrt{5}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\rightarrow(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}a^3=$$

$$\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)=s^3-3.4Rrs\Rightarrow\sum_{cyc}a^3=s^3-12Rrs\rightarrow(3)$$

$$\text{Now, } a^3+b^3+c^3+\sqrt{5}abc\ge 3+\sqrt{5}\Leftrightarrow\sum_{cyc}a^3-3\ge\sqrt{5}(1-abc)$$

$$\overset{a+b+c=3}{\Leftrightarrow}\sum_{cyc}a^3-\frac{\left(\sum_{cyc}a\right)^3}{9}\ge\sqrt{5}\left(\frac{\left(\sum_{cyc}a\right)^3}{27}-abc\right)$$

$$\Leftrightarrow 3\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)\overset{(*)}{\ge}\sqrt{5}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\text{ and }\because\sqrt{5}<\frac{9}{4}\text{ and}$$

$$\left(\sum_{cyc}a\right)^3-27abc\overset{A-G}{\ge}0\therefore\text{ in order to prove }(*),\text{ it suffices to prove:}$$

$$3\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)\ge\frac{9}{4}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}$$

$$4(9(s^3-12Rrs)-s^3)\ge 3(s^3-27r^2s)\Leftrightarrow 29s^2\overset{(**)}{\ge}432Rr-81r^2$$

$$\text{Again, }29s^2\overset{\text{Gerretsen}}{\ge}29(16Rr-5r^2)\overset{?}{\ge}432Rr-81r^2\Leftrightarrow 32Rr\overset{?}{\ge}64r^2\rightarrow\text{true}$$

$$\text{via Euler}\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore a^3+b^3+c^3+\sqrt{5}abc\ge 3+\sqrt{5}$$

$$\forall a,b,c>0\,\middle|\,a+b+c=3,\text{"="}\text{ iff }a=b=c=1\text{ (QED)}$$