ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $ab(a^2 + b^2) \geq 2$, then prove that:

$$a^3 + b^3 \geq 2$$

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Let $a = r\cos\theta, b = r\sin\theta, r > 0, 0 < \theta < \frac{\pi}{2}$

$$ab(a^2 + b^2) \geq 2 \Rightarrow r^4 \sin\theta\cos\theta \geq 2 \Rightarrow r^4 \geq \frac{4}{\sin(2\theta)} \geq 4 \Rightarrow r \geq \sqrt{2}$$

Now, $a^3 + b^3 = r^3(\cos^3\theta + \sin^3\theta)$. Let $f(\theta) = \cos^3\theta + \sin^3\theta, 0 < \theta < \frac{\pi}{2}$

$$f'(\theta) = 3\cos\theta\sin\theta(\sin\theta - \cos\theta)$$

$$f'(\theta) < 0 \text{ if } 0 < \theta < \frac{\pi}{4}$$

$$= 0 \text{ if } \theta = \frac{\pi}{4}$$

$$> 0 \text{ if } \frac{\pi}{4} < \theta < \frac{\pi}{2}$$

$$\therefore \min(f(\theta)) = f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Thus, $a^3 + b^3 \geq 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$

Equality when $a = b = 1$.