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If $a, b, c \in [0,1]$, then prove that :

$$\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2$$

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$\boxed{\text{Case 1}}$ $a=b=c=0$ and then : $\text{LHS}=3.\frac{1}{2}=\frac{3}{2}<2\ \therefore \sum_{cyc}\frac{a+1}{b+2}<2$

$\boxed{\text{Case 2}}$ Exactly 2 variables equal zero and WLOG we may assume : $b=c=0$ and then : $\text{LHS}=\frac{a+1}{2}+\frac{1}{2}+\frac{1}{a+2}\overset{?}{<}2\Leftrightarrow\frac{(a+1)(a+2)+2}{2(a+2)}\overset{?}{<}\frac{3}{2}$

$$\Leftrightarrow a^2+3a+4\overset{?}{<}3a+6\Leftrightarrow a\overset{?}{<}\sqrt{2}\to\text{true}\because 0\le a\le 1<\sqrt{2}$$

$$\therefore\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}<2$$

$\boxed{\text{Case 3}}$ Exactly 1 variables equals zero and WLOG we may assume $a=0$ and then : $\text{LHS}=\frac{1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{2}$

$$=\frac{2c+4+2(b+1)(b+2)+(b+2)(c+2)(c+1)}{2(b+2)(c+2)}\overset{?}{\le}2\Leftrightarrow$$

$bc+4\overset{?}{\ge}bc^2+2b^2+2c^2\to\text{true}\because 0\le b,c\le 1\Rightarrow bc\ge bc^2, 2\ge 2b^2$ and $2\ge 2c^2$

$$\therefore\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2,''='' \text{ iff } (a=0, b=c=1) \text{ and permutations}$$

$\boxed{\text{Case 4}}$ $a,b,c>0$ and then : $\left(\frac{1}{a}-1\right),\left(\frac{1}{b}-1\right),\left(\frac{1}{c}-1\right)\ge 0$ and denoting $\frac{1}{a}-1$ $=x, \frac{1}{b}-1=y, \frac{1}{c}-1=z$, we get : $a=\frac{1}{x+1}, b=\frac{1}{y+1}, c=\frac{1}{z+1}$

Now, following simplification, $\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2$ is equivalent to :

$$\sum_{cyc}ab+4+2abc\ge\sum_{cyc}ab^2+2\sum_{cyc}a^2$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{1}{x+1}.\frac{1}{y+1}\right)+4+2\left(\frac{1}{x+1}.\frac{1}{y+1}.\frac{1}{z+1}\right)\ge$$

$$\sum_{cyc}\left(\frac{1}{x+1}.\left(\frac{1}{y+1}\right)^2\right)+2\sum_{cyc}\left(\frac{1}{x+1}\right)^2$$

$$\Leftrightarrow\frac{\sum_{cyc}x+3+4\prod_{cyc}(x+1)+2}{\prod_{cyc}(x+1)}\ge$$

$$\frac{\sum_{cyc}\left((x+1)^2(y+1)\right)+2\sum_{cyc}((x+1)^2(y+1)^2)}{\prod_{cyc}(x+1)^2}$$

$$\Leftrightarrow 4x^2y^2z^2+8xyz\left(\sum_{cyc}xy\right)+2\sum_{cyc}x^2y^2+17xyz\left(\sum_{cyc}x\right)+4\sum_{cyc}x^2y+5\sum_{cyc}xy^2$$

$$+40xyz+13\sum_{cyc}xy+3\sum_{cyc}x\geq 0 \to \text{true} \because x,y,z\geq 0,''='' \text{ iff } x=y=z=0 \Rightarrow \text{iff}$$

$$a=b=c=1 \therefore \text{combining } all\ cases, \frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\leq 2\ \forall\, a,b,c\in[0,1],$$

$$''='' \text{ iff } (a=0,b=c=1) \text{ or } (b=0,a=c=1) \text{ or } (c=0,a=b=1)$$

$$\text{or } (a=b=c=1)\ (\text{QED})$$