ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $a^3b^3 + b^3c^3 + c^3a^3 = (abc)^4$ then:

$$\frac{1}{a^6} + \frac{1}{b^6} + \frac{1}{c^6} \geq 1$$

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Solution by Tapas Das-India

$$a^3b^3 + b^3c^3 + c^3a^3 = (abc)^4$$

$$(abc)^4 \overset{Am-Gm}{\geq} 3(abc)^2 \text{ or } (abc)^2 \geq 3$$

$$\frac{1}{a^6} + \frac{1}{b^6} + \frac{1}{c^6} = \sum \left(\frac{1}{a^3}\right)^2 \geq \frac{1}{3}\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right)^2 =$$

$$= \frac{1}{3}\left(\frac{\sum a^3b^3}{(abc)^3}\right)^2 = \frac{1}{3}\left(\frac{(abc)^4}{(abc)^3}\right)^2 = \frac{1}{3}(abc)^2 \geq \frac{1}{3}.3 = 1$$

$$Equality\ for\ a = b = c = \sqrt[6]{3}$$