ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \left[\frac{1}{2}, 1\right]$, then prove that :

$$a^5b + ab^5 + \frac{6}{a^2+b^2} - 3(a+b) \geq -1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a, b \in \left[\frac{1}{2}, 1\right] \Rightarrow (a-1), (b-1) \leq 0 \Rightarrow (a-1)(b-1) \geq 0$$
$$\Rightarrow ab \geq a+b-1 \rightarrow (1)$$

$$\text{Now, } a^5b + ab^5 + \frac{6}{a^2+b^2} - 3(a+b) + 1$$
$$= ab(a^4+b^4) + \frac{6}{(a+b)^2 - 2ab} - 3(a+b) + 1$$
$$\overset{\text{Holder}}{\geq} \frac{ab(a+b)^4}{8} + \frac{6}{(a+b)^2 - 2ab} - 3(a+b) + 1$$
$$\overset{\text{via (1)}}{\geq} \frac{(a+b-1)(a+b)^4}{8} + \frac{6}{(a+b)^2 - 2(a+b-1)} - 3(a+b) + 1$$
$$= \frac{(t-1)t^4}{8} + \frac{6}{t^2-2t+2} - 3t + 1 \ (t = a+b)$$
$$= \frac{(t-1)(t^2-2t+2)t^4 + 48 - 24t(t^2-2t+2) + 8(t^2-2t+2)}{8(t^2-2t+2)}$$
$$= \frac{t^7 - 3t^6 + 4t^5 - 2t^4 - 24t^3 + 56t^2 - 64t + 64}{8(t^2-2t+2)}$$
$$= \frac{(t-2)^2\left(t^5 + t^4 + 4t^3 + 10t^2 + 16\right)}{8(t^2-2t+2)} \geq 0$$
$$\because t = a+b \geq \frac{1}{2} + \frac{1}{2} > 0$$
$$\text{and } t^2 - 2t + 2 = (t-1)^2 + 1 > 0 \therefore a^5b + ab^5 + \frac{6}{a^2+b^2} - 3(a+b) \geq -1$$
$$\forall\, a, b \in \left[\frac{1}{2}, 1\right], '' ='' \text{ iff } a = b = 1 \text{ (QED)}$$