ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} = 3$, then prove that :

$$\frac{2}{3 + ab + \mathrm{bc} + \mathrm{c}a} + \sqrt[3]{\frac{abc}{(1+a)(1+\mathrm{b})(1+\mathrm{c})}} \le \frac{5}{6}$$

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$$\frac{5}{6} - \frac{2}{3+\sum_{cyc} ab} = \frac{15 + 5\sum_{cyc} ab - 12}{6\left(3+\sum_{cyc} ab\right)} \Rightarrow \frac{5}{6} - \frac{2}{3+\sum_{cyc} ab}$$

$$= \frac{3+5x}{6(3+x)} \to (1)\ \left(x = \sum_{cyc} ab\right)$$

$$\text{Now,} \frac{abc}{(1+a)(1+b)(1+c)} = \frac{abc}{1 + abc + \sum_{cyc} ab + \sum_{cyc} a}$$

$$= \frac{abc + 1 + \sum_{cyc} ab + \sum_{cyc} a - \left(1 + \sum_{cyc} ab + \sum_{cyc} a\right)}{1 + abc + \sum_{cyc} ab + \sum_{cyc} a}$$

$$= 1 - \frac{1 + \sum_{cyc} ab + \sum_{cyc} a}{1 + abc + \sum_{cyc} ab + \sum_{cyc} a} \le 1 - \frac{1 + \sum_{cyc} ab + \sum_{cyc} a}{1 + \frac{\left(\sum_{cyc} ab\right)^2}{9} + \sum_{cyc} ab + \sum_{cyc} a}$$

$$\left(\because 3abc\left(\sum_{cyc} a\right) \le \left(\sum_{cyc} ab\right)^2 \overset{a+b+c=3}{\Rightarrow} abc \le \frac{1}{9}\left(\sum_{cyc} ab\right)^2\right)$$

$$= \frac{\frac{\left(\sum_{cyc} ab\right)^2}{9}}{1 + \frac{\left(\sum_{cyc} ab\right)^2}{9} + \sum_{cyc} ab + \sum_{cyc} a} \overset{a+b+c=3}{=} \frac{x^2}{9 + x^2 + 9x + 27}$$

$$\Rightarrow \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}} \le \sqrt[3]{\frac{x^2}{36 + x^2 + 9x}} \overset{?}{\le} \frac{3+5x}{6(3+x)}$$

$$\Leftrightarrow \frac{(3+5x)^3}{216(3+x)^3} \overset{?}{\ge} \frac{x^2}{36 + x^2 + 9x} \Leftrightarrow (36 + x^2 + 9x)(3+5x)^3 \overset{?}{\ge} 216x^2(3+x)^3$$

$$\Leftrightarrow -91x^5 - 594x^4 + 828x^3 + 3510x^2 + 5103x + 972 \overset{?}{\ge} 0$$

$$\Leftrightarrow (3-x)\left(91x^4 + 867x^3 + 1773x^2 + 1809x + 324\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because x = \sum_{cyc} ab > 0 \text{ and } 3x \le \left(\sum_{cyc} a\right)^2 = 9 \Rightarrow 3 - x \ge 0$$

$$\therefore \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}} \le \frac{3+5x}{6(3+x)} \overset{\text{via }(1)}{=} \frac{5}{6} - \frac{2}{3+\sum_{cyc} ab}$$

$$\Rightarrow \frac{2}{3 + ab + bc + ca} + \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}} \le \frac{5}{6}$$

$$\forall\, a, b, c > 0 \,\middle|\, a + b + c = 3, ''='' \text{ iff } a = b = c = 1 \text{ (QED)}$$