

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$ then:

$$xyz(x+y+z)(x+1)(y+1)(z+1) \geq 1296$$

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$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1 \quad (1)$$

$$\text{or } x + y + z + 2 = xyz \quad (2)$$

$$\text{Now } [(x+1) + (y+1) + (z+1)] \left[\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \right] \stackrel{C-S}{\geq} 9 \text{ or}$$

$$(x+y+z+3) \stackrel{(1)}{\geq} 9 \text{ or } x+y+z \geq 6 \quad (2)$$

$$\text{Now from (2) } xyz \geq 6 + 2 = 8 \quad (3)$$

$$\text{and } xy + yz + zx \stackrel{AM-GM}{\geq} 3(xyz)^{\frac{2}{3}} \stackrel{(3)}{\geq} 3 \cdot (8)^{\frac{2}{3}} = 12 \quad (4) \text{ then}$$

$$\begin{aligned} & xyz(x+y+z)(x+1)(y+1)(z+1) = \\ & = xyz(x+y+z)(1+x+y+z+xy+yz+zx+xyz) \stackrel{(2)\&(3)\&(4)}{\geq} \\ & \geq 8 \cdot 6(1+6+12+8) = 1296 \end{aligned}$$

Equality for $x = y = z = 2$