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If $a, b, c > 0$ and $a^5 + b^5 + c^5 = 3$, then prove that :

$$\frac{a^6}{bc} + \frac{b^6}{ca} + \frac{c^6}{ab} \geq 3$$

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$$3 = \sum_{cyc} a^5 \overset{\text{Holder}}{\geq} \frac{1}{81}\left(\sum_{cyc} a\right)^5 \Rightarrow \sum_{cyc} a \leq 3 \Rightarrow \sum_{cyc} ab \leq \frac{1}{3}\left(\sum_{cyc} a\right)^2 \leq \frac{9}{3}$$

$$\therefore \sum_{cyc} ab \leq 3 \rightarrow (1)$$

Now, WLOG we may assume $a \geq b \geq c$ and then : $a^6 \geq b^6 \geq c^6$ and

$$\frac{1}{bc} \geq \frac{1}{ca} \geq \frac{1}{ab} \therefore \frac{a^6}{bc} + \frac{b^6}{ca} + \frac{c^6}{ab} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} a^6\right)\left(\sum_{cyc} \frac{1}{bc}\right)$$

$$\overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{Bergstrom}}}{\geq} \frac{1}{9}\left(\sum_{cyc} a^5\right)\left(\sum_{cyc} a\right) \cdot \frac{9}{\sum_{cyc} ab} \geq \frac{3 \cdot \sqrt{3\sum_{cyc} ab}}{\sum_{cyc} ab}\left(\because \sum_{cyc} a^5 = 3\right) = \frac{3\sqrt{3}}{\sqrt{\sum_{cyc} ab}}$$

$$\overset{\text{via } (1)}{\geq} \frac{3\sqrt{3}}{\sqrt{3}} \therefore \frac{a^6}{bc} + \frac{b^6}{ca} + \frac{c^6}{ab} \geq 3 \ \forall\, a, b, c > 0 \,\middle|\, a^5 + b^5 + c^5 = 3,$$

$'' ='' $ iff $a = b = c = 1$ (QED)