ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\frac{2(a^5+b^5+c^5+a^2+b^2+c^2)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}\geq 5$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2(a^5+b^5+c^5+a^2+b^2+c^2)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}$$

$$=\frac{\sum_{cyc}a^5}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^5}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^2}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^2}{\sum_{cyc}a^3}+\frac{9}{\left(\sum_{cyc}a\right)^2}\overset{A-G}{\geq}5.\sqrt[5]{\frac{9\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a^2\right)^2}{\left(\sum_{cyc}a^3\right)^4\left(\sum_{cyc}a\right)^2}}$$

$$\geq 5.\sqrt[5]{\frac{\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a\right)^4}{\left(\sum_{cyc}a^3\right)^4\left(\sum_{cyc}a\right)^2}}\left(\because\sum_{cyc}a^2\geq\frac{1}{3}\left(\sum_{cyc}a\right)^2\right)=5.\sqrt[5]{\frac{\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a^3\right)^4}}$$

$$\geq 5.\sqrt[5]{\frac{\left(\sum_{cyc}a^3\right)^4}{\left(\sum_{cyc}a^3\right)^4}}\left(\because\left(\sum_{cyc}a^5\right)\left(\sum_{cyc}a\right)\overset{\text{Reverse CBS}}{\geq}\left(\sum_{cyc}a^3\right)^2\right)$$

$$\therefore\frac{2(a^5+b^5+c^5+a^2+b^2+c^2)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}\geq 5$$

$\forall\, a, b, c > 0$, " = " iff $a = b = c$ (QED)