ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that:

$$\frac{a^3+5}{a^3(b+c)}+\frac{b^3+5}{b^3(c+a)}+\frac{c^3+5}{c^3(a+b)} \geq 9$$

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$$\sum_{cyc}\frac{a^3+5}{a^3(b+c)} \overset{abc=1}{=} \sum_{cyc}\frac{1}{b+c}+5\sum_{cyc}\frac{b^3c^3}{b+c} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Holder}}}{\geq} \frac{9}{2\sum_{cyc}a}+\frac{5\left(\sum_{cyc}ab\right)^3}{6\sum_{cyc}a}$$

$$\geq \frac{9}{2\sum_{cyc}a}+\frac{15abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)}{6\sum_{cyc}a} \overset{abc=1}{=} \frac{9}{2\sum_{cyc}a}+\frac{5\left(\sum_{cyc}ab\right)}{2}$$

$$\geq \frac{9}{2\sum_{cyc}a}+\frac{5.\sqrt{3abc\left(\sum_{cyc}a\right)}}{2} \overset{abc=1}{=} \frac{9}{2\left(\frac{t^2}{3}\right)}+\frac{5t}{2}\left(t=\sqrt{3\sum_{cyc}a}\right) \overset{?}{\geq} 9$$

$$\Leftrightarrow 5t^3-18t^2+27 \overset{?}{\geq} 0 \Leftrightarrow (t-3)\big((t-3)(5t+12)+27\big) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t=\sqrt{3\sum_{cyc}a} \overset{A-G}{\geq} \sqrt{9.\sqrt[3]{abc}} \overset{abc=1}{=} 3 \therefore \frac{a^3+5}{a^3(b+c)}+\frac{b^3+5}{b^3(c+a)}+\frac{c^3+5}{c^3(a+b)} \geq 9$$

$$\forall\, a,b,c>0 \,\big|\, abc=1, \text{''}=\text{''} \text{ iff } a=b=c=1 \text{ (QED)}$$