ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = \frac{1}{6}$, then prove that :

$$\frac{a^2}{(1+a)^2} + \frac{4b^2}{(1+2b)^2} + \frac{36c^3}{(1+3c)^3} \geq \frac{2}{3}$$

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$$\text{We consider } f(t) = \frac{4t^3}{3(t+1)^3} - \frac{t^2}{(t+1)^2}\ \forall\, t > 0; \textit{then}:$$

$$f'(t) = \frac{2t(t-1)}{(t+1)^4} \geq 0\ \forall\, t \geq 1 \textit{ and } \leq 0\ \forall\, t \leq 1 \therefore f(t) \text{ is} \uparrow \text{on } [1,\infty) \textit{ and } f(t) \text{ is} \downarrow \text{on}$$

$$(0,1] \therefore f(t) \text{ is minimum } \textit{at } t = 1 \Rightarrow f(t) \geq f(1) = -\frac{1}{12}$$

$$\therefore \frac{4t^3}{3(t+1)^3} - \frac{t^2}{(t+1)^2} \geq -\frac{1}{12}\ \forall\, t > 0 \to (1)$$

$$\text{Let } a = x, 2b = y \textit{ and } 3c = z; \text{then}: xyz = 1 \left(\because abc = \frac{1}{6}\right)$$

$$\textit{and } \frac{a^2}{(1+a)^2} + \frac{4b^2}{(1+2b)^2} + \frac{36c^3}{(1+3c)^3} = \frac{x^2}{(x+1)^2} + \frac{y^2}{(1+y)^2} + \frac{4z^3}{3(1+z)^3}$$

$$= \frac{x^2}{(x+1)^2} + \frac{y^2}{(1+y)^2} + \frac{z^2}{(z+1)^2} + \frac{4z^3}{3(1+z)^3} - \frac{z^2}{(z+1)^2} \overset{\text{via }(1)}{\geq}$$

$$\frac{x^2}{(x+1)^2} + \frac{y^2}{(1+y)^2} + \frac{z^2}{(z+1)^2} - \frac{1}{12} \overset{?}{\geq} \frac{2}{3} \Leftrightarrow \frac{x^2}{(x+1)^2} + \frac{y^2}{(1+y)^2} + \frac{z^2}{(z+1)^2} \boxed{\underset{(*)}{\overset{?}{\geq}}} \frac{3}{4}$$

$$\text{Now, } \frac{x^2}{(x+1)^2} + \frac{y^2}{(y+1)^2} + \frac{z^2}{(z+1)^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}} x\right)^2}{\sum_{\text{cyc}} x^2 + 2\sum_{\text{cyc}} x + 3}$$

$$= \frac{\sum_{\text{cyc}} x^2 + 2\sum_{\text{cyc}} xy}{\sum_{\text{cyc}} x^2 + 2\sum_{\text{cyc}} x + 3} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow 4\sum_{\text{cyc}} x^2 + 8\sum_{\text{cyc}} xy \overset{?}{\geq} 3\sum_{\text{cyc}} x^2 + 6\sum_{\text{cyc}} x + 9$$

$$\Leftrightarrow \sum_{\text{cyc}} x^2 + 8\sum_{\text{cyc}} xy - 6\sum_{\text{cyc}} x - 9 \boxed{\underset{(**)}{\overset{?}{\geq}}} 0$$

$$\text{Now, LHS of } (**) \geq \frac{1}{3}\left(\sum_{\text{cyc}} x\right)^2 + 8.\sqrt{3xyz\sum_{\text{cyc}} x} - 6\sum_{\text{cyc}} x - 9$$

$$\overset{xyz=1}{\geq} \frac{1}{3}\left(\sum_{\text{cyc}} x\right)^2 + 8.\sqrt{3\sum_{\text{cyc}} x} - 6\sum_{\text{cyc}} x - 9$$

$$=\frac{u^4}{9}+24u-18.\frac{u^2}{3}-27\left(u=\sqrt{3\sum_{cyc}x}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow u^4-54u^2+216u-243\overset{?}{\ge}0\Leftrightarrow(u+9)(u-3)^3\overset{?}{\ge}0\to\text{true}\because u=\sqrt{3\sum_{cyc}x}$$

$$\overset{A-G}{\ge}\sqrt{9.\sqrt[3]{xyz}}\overset{xyz=1}{=}3\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a^2}{(1+a)^2}+\frac{4b^2}{(1+2b)^2}+\frac{36c^3}{(1+3c)^3}$$

$$\ge\frac{2}{3}\ \forall a,b,c>0\,\Big|\,abc=\frac{1}{6},''=''\text{ iff } a=1,b=\frac{1}{2},c=\frac{1}{3}\ (\text{QED})$$