ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0$, *then prove that* :

$$\frac{a^2}{(a+\mathrm{b})^2}+\frac{\mathrm{b}^2}{(\mathrm{b}+\mathrm{c})^2}+\frac{4\mathrm{c}^3}{3(\mathrm{c}+a)^3}\geq\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4c^3}{3(c+a)^3}-\frac{c^2}{(c+a)^2}=\frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\ \left(t=\frac{c}{a}\right)\text{ and we consider}$$

$$f(t)=\frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\ \forall\, t>0;\ then:$$

$$f'(t)=\frac{2t(t-1)}{(t+1)^4}\geq 0\ \forall\, t\geq 1\ and\ \leq 0\ \forall\, t\leq 1$$

$$\therefore f(t)\text{ is }\uparrow\text{ on }[1,\infty)\ and\ f(t)\text{ is }\downarrow\text{ on }(0,1]\ \therefore f(t)\text{ is minimum }at\ t=1\Rightarrow f(t)\geq f(1)$$

$$=-\frac{1}{12}\ \therefore\ \frac{4c^3}{3(c+a)^3}-\frac{c^2}{(c+a)^2}\geq-\frac{1}{12}\to(1)$$

$$\text{Now,}\ \frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}=\frac{x^2}{(x+1)^2}+\frac{y^2}{(y+1)^2}+\frac{z^2}{(z+1)^2}$$

$$\left(x=\frac{a}{b},y=\frac{b}{c},z=\frac{c}{a}\right)\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x^2+2\sum_{cyc}x+3}=\frac{\sum_{cyc}x^2+2\sum_{cyc}xy}{\sum_{cyc}x^2+2\sum_{cyc}x+3}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow 4\sum_{cyc}x^2+8\sum_{cyc}xy\overset{?}{\geq}3\sum_{cyc}x^2+6\sum_{cyc}x+9$$

$$\Leftrightarrow\sum_{cyc}x^2+8\sum_{cyc}xy-6\sum_{cyc}x-9\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, LHS of }(*)\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3xyz\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$\overset{xyz=1}{\geq}\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$=\frac{u^4}{9}+24u-18.\frac{u^2}{3}-27\ \left(u=\sqrt{3\sum_{cyc}x}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow u^4-54u^2+216u-243\overset{?}{\geq}0\Leftrightarrow(u+9)(u-3)^3\overset{?}{\geq}0\to\text{true}\ \because u=\sqrt{3\sum_{cyc}x}$$

$$=\sqrt{3\sum_{cyc}\frac{a}{b}}\overset{A-G}{\geq}3\Rightarrow(*)\text{ is true}\Rightarrow\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}\geq\frac{3}{4}\to(2)$$

$$\therefore(1)+(2)\Rightarrow\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{4c^3}{3(c+a)^3}\geq\frac{3}{4}-\frac{1}{12}=\frac{2}{3}$$

$$\forall\, a,b,c>0,\ ''=''\ \text{iff } a=b=c\ (\text{QED})$$