ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c \in [1,2]$, then prove that :

$$a^2+b^2+c^2+3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)$$

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$$\forall\, a,b,c>0, \sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 2\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 2\sum_{cyc} ab$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^3 + 27a^2b^2c^2 + 9\left(\sum_{cyc} a^2\right)^2 . \sqrt[3]{a^2b^2c^2}$$

$$+27\left(\sum_{cyc} a^2\right) . \sqrt[3]{a^4b^4c^4} \boxed{\overset{?}{\underset{(*)}{\geq}}} 8\left(\sum_{cyc} ab\right)^3$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$$

$$\therefore abc = r^2 s \to (2) \text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y) \Rightarrow$$

$$\sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via } (1),(3)}{=} s^2 - 2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

Via (1), (2) and (4), LHS of $(*) \geq$

$$(s^2-8Rr-2r^2)^3 + 27r^4s^2 + 9(s^2-8Rr-2r^2)^2 . \sqrt[3]{r^4s^2}$$

$$+27(s^2-8Rr-2r^2) . \sqrt[3]{r^8s^4} \overset{\text{Mitrinovic}}{\geq}$$

$$(s^2-8Rr-2r^2)^3 + 27r^4s^2 + 9(s^2-8Rr-2r^2)^2 . \sqrt[3]{r^4 . 27r^2}$$

$$+27(s^2-8Rr-2r^2) . \sqrt[3]{r^8 . 729r^4}$$

$$= (s^2-8Rr-2r^2)^3 + 27r^4s^2 + 27r^2(s^2-8Rr-2r^2)^2 + 243r^4(s^2-8Rr-2r^2)$$

$$\overset{?}{\geq} 8\left(\sum_{cyc} ab\right)^3 \overset{\text{via } (3)}{=} 8(4Rr+r^2)^3$$

$$\Leftrightarrow s^6 - (24Rr - 21r^2)s^4 + r^2(192R^2 - 336Rr + 174r^2)s^2$$

$$-r^3(1024R^3 - 960R^2r + 1272Rr^2 + 394r^3) \boxed{\overset{?}{\underset{(**)}{\geq}}} 0 \text{ and}$$

$$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$
$$\text{it suffices to prove : LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^3$$
$$\Leftrightarrow (8R + 2r)s^4 - r(192R^2 - 48Rr - 33r^2)s^2$$
$$+r^2(1024R^3 - 960R^2r - 24Rr^2 - 173r^3) \boxed{\overset{(***)}{\geq}} 0 \text{ and } \because$$
$$(8R + 2r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$
$$\text{it suffices to prove : LHS of } (***) \geq (8R + 2r)(s^2 - 16Rr + 5r^2)^2$$
$$\Leftrightarrow (64R^2 + 32Rr + 13r^2)s^2 \boxed{\overset{(****)}{\geq}} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3)$$
$$\text{Finally}, (64R^2 + 32Rr + 13r^2)s^2 \overset{\text{Gerretsen}}{\geq} (64R^2 + 32Rr + 13r^2)(16Rr - 5r^2)$$
$$\overset{?}{\geq} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3) \Leftrightarrow 144r^3(R - 2r) \overset{?}{\geq} 0 \to \text{true via Euler}$$
$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \forall a, b, c > 0, \sum_{\text{cyc}} a^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2\sum_{\text{cyc}} ab$$
$$\therefore a^2 + b^2 + c^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2(ab + bc + ca)\ \forall a, b, c \in [1, 2],$$
$$" = " \text{ iff } a = b = c \text{ (QED)}$$