ROMANIAN MATHEMATICAL MAGAZINE

$$If\ x,y,z>0\ ,x^2+y^2+z^2=3\ then$$
$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}-\frac{1}{x+y+z}\ge\frac{17}{3}$$

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$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3}\overset{AM-HM}{\ge}\frac{3}{x+y+z}\ or$$
$$\frac{1}{x+y+z}\le\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ (1)$$

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\ge\frac{3(x^2+y^2+z^2)}{x+y+z}\ (2)$$

$Proof$:

$$(x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\ge 3(x^2+y^2+z^2)or$$
$$\sum\frac{x^3}{y}+\sum\frac{xy^2}{z}\ge 2\sum x^2\ \ or$$

$$\sum\frac{x^4}{yx}+\sum\frac{x^2y^2}{zx}\ge 2\sum x^2\ \ or$$

$$\frac{(\sum x^2)^2}{\sum xy}+\frac{(\sum xy)^2}{\sum xy}\ge 2\sum x^2\ \ (Bergstrom)$$

$$or\ \left(\sum x^2\right)^2+\left(\sum xy\right)^2-2\left(\sum xy\right)\left(\sum x^2\right)\ge 0\ or$$

$$\left(\sum x^2-\sum xy\right)^2\ge 0\ \ (Proof\ complete)$$

$$(x+y+z)^2\overset{CBS}{\le}3(x^2+y^2+z^2)\ or$$
$$(x+y+z)\le\sqrt{3(x^2+y^2+z^2)}\ (3)$$

$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}-\frac{1}{x+y+z}=$$

$$=\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\left(\frac{1}{x+y+z}\right)\ge$$

$$\overset{(1)\&(2)}{\ge}\frac{3(x^2+y^2+z^2)}{x+y+z}+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$=\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{8}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge$$

$$\overset{Bergstrom}{\ge}\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{8}{9}\frac{(1+1+1)^2}{x+y+z}\overset{(3)}{\ge}$$

$$\ge\frac{3(x^2+y^2+z^2)}{\sqrt{3(x^2+y^2+z^2)}}+\frac{8}{\sqrt{3(x^2+y^2+z^2)}}=$$

$$=\frac{3.3}{\sqrt{3.3}}+\frac{8}{\sqrt{3.3}}(since\ \ x^2+y^2+z^2=3)=3+\frac{8}{3}=\frac{17}{3}$$

$$Equality\ holds\ for\ x=y=z=1$$