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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a\mathrm{bc} = 1$, then prove that :

$$a^3 + \mathrm{b}^3 + \mathrm{c}^3 + 6 \ge (a + \mathrm{b} + \mathrm{c})^2$$

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Assigning $\mathrm{b} + \mathrm{c} = x, \mathrm{c} + a = \mathrm{y}, a + \mathrm{b} = \mathrm{z} \Rightarrow x + \mathrm{y} - \mathrm{z} = 2\mathrm{c} > 0$,
$\mathrm{y} + \mathrm{z} - x = 2a > 0$ and $\mathrm{z} + x - \mathrm{y} = 2\mathrm{b} > 0 \Rightarrow x + \mathrm{y} > \mathrm{z}, \mathrm{y} + \mathrm{z} > x, \mathrm{z} + x > \mathrm{y}$
$\Rightarrow x, \mathrm{y}, \mathrm{z}$ form sidesof a triangle with semiperimeter, circumradius and inradius
$= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say); so $2\sum_{\mathrm{cyc}} a = \sum_{\mathrm{cyc}} x = 2\mathrm{s} \Rightarrow \sum_{\mathrm{cyc}} a = \mathrm{s} \to (1)$
$\Rightarrow a = \mathrm{s} - x, \mathrm{b} = \mathrm{s} - \mathrm{y}, \mathrm{c} = \mathrm{s} - \mathrm{z} \therefore a\mathrm{bc} = \mathrm{r}^2\mathrm{s} \to (2)$

Such substitutions $\Rightarrow \sum_{\mathrm{cyc}} a^3 = \left(\sum_{\mathrm{cyc}} a\right)^3 - 3(a + \mathrm{b})(\mathrm{b} + \mathrm{c})(\mathrm{c} + a) \overset{\text{via (1)}}{=} \mathrm{s}^3 - 3x\mathrm{yz}$
$= \mathrm{s}^3 - 12\mathrm{Rrs} \therefore \sum_{\mathrm{cyc}} a^3 = \mathrm{s}^3 - 12\mathrm{Rrs} \to (3)$

Now, $a^3 + \mathrm{b}^3 + \mathrm{c}^3 + 6 \ge 2(a + \mathrm{b} + \mathrm{c})^2 \overset{a\mathrm{bc}=1}{\Leftrightarrow} \sum_{\mathrm{cyc}} a^3 + 6a\mathrm{bc} \ge \sqrt[3]{a\mathrm{bc}}\left(\sum_{\mathrm{cyc}} a\right)^2$
$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} (\mathrm{s}^3 - 12\mathrm{Rrs} + 6\mathrm{r}^2\mathrm{s})^3 \ge \mathrm{r}^2\mathrm{s}^7 \Leftrightarrow (\mathrm{s}^2 - 12\mathrm{Rr} + 6\mathrm{r}^2)^3 \overset{(*)}{\ge} \mathrm{r}^2\mathrm{s}^4$ and
$\because (\mathrm{s}^2 - 16\mathrm{Rr} + 5\mathrm{r}^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove $(*)$, it suffices to prove :
LHS of $(*) \ge (\mathrm{s}^2 - 16\mathrm{Rr} + 5\mathrm{r}^2)^3 \Leftrightarrow (12\mathrm{R} + 2\mathrm{r})\mathrm{s}^4 - \mathrm{r}(336\mathrm{R}^2 - 48\mathrm{Rr} - 33\mathrm{r}^2)\mathrm{s}^2$
$+\mathrm{r}^2(2368\mathrm{R}^3 - 1248\mathrm{R}^2\mathrm{r} - 96\mathrm{Rr}^2 + 91\mathrm{r}^3) \overset{(**)}{\ge} 0$ and
$\because (12\mathrm{R} + 2\mathrm{r})(\mathrm{s}^2 - 16\mathrm{Rr} + 5\mathrm{r}^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove $(**)$,
it suffices to prove : LHS of $(**) \ge (12\mathrm{R} + 2\mathrm{r})(\mathrm{s}^2 - 16\mathrm{Rr} + 5\mathrm{r}^2)^2$
$\Leftrightarrow (48\mathrm{R}^2 - 8\mathrm{Rr} + 13\mathrm{r}^2)\mathrm{s}^2 \overset{(***)}{\ge} \mathrm{r}(704\mathrm{R}^3 - 160\mathrm{R}^2\mathrm{r} + 76\mathrm{Rr}^2 - 41\mathrm{r}^3)$
Finally, $(48\mathrm{R}^2 - 8\mathrm{Rr} + 13\mathrm{r}^2)\mathrm{s}^2 \overset{\text{Gerretsen}}{\ge} (48\mathrm{R}^2 - 8\mathrm{Rr} + 13\mathrm{r}^2)(16\mathrm{Rr} - 5\mathrm{r}^2) \overset{?}{\ge}$
$\mathrm{r}(704\mathrm{R}^3 - 160\mathrm{R}^2\mathrm{r} + 76\mathrm{Rr}^2 - 41\mathrm{r}^3) \Leftrightarrow 16\mathrm{t}^3 - 52\mathrm{t}^2 + 43\mathrm{t} - 6 \overset{?}{\ge} 0$
$\Leftrightarrow (\mathrm{t} - 2)(6\mathrm{t}^2 + 10\mathrm{t}(\mathrm{t} - 2) + 3) \overset{?}{\ge} 0 \to$ true $\because \mathrm{t} \overset{\text{Euler}}{\ge} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$
is true $\therefore a^3 + \mathrm{b}^3 + \mathrm{c}^3 + 6 \ge (a + \mathrm{b} + \mathrm{c})^2 \;\forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\big|\, a\mathrm{bc} = 1$,
" = " iff $a = \mathrm{b} = \mathrm{c} = 1$ (QED)