ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3} \le \frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}$$

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$$\frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}=$$

$$=\frac{2a^2}{a+b}+\frac{2b^2}{b+c}+\frac{2c^2}{c+a}+2\left(\frac{(a^2-b^2)+b^2}{a+b}+\frac{(b^2-c^2)+c^2}{b+c}+\frac{(c^2-a^2)+a^2}{c+a}\right)=$$

$$=\frac{2a^2}{a+b}+\frac{2b^2}{b+c}+\frac{2c^2}{c+a}+2\sum_{cyc}(a-b)+\frac{2b^2}{a+b}+\frac{2c^2}{b+c}+\frac{2a^2}{c+a}$$

$$\therefore \frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}=\frac{2a^2+2b^2}{a+b}+\frac{2b^2+2c^2}{b+c}+\frac{2c^2+2a^2}{c+a} \to (1)$$

$$\text{Now, } \frac{2b^2+2c^2}{b+c} \overset{?}{\ge} \sqrt[3]{4b^3+4c^3} \Leftrightarrow (2b^2+2c^2)^3 \overset{?}{\ge} (4b^3+4c^3)(b+c)^3$$

$$\Leftrightarrow t^6-3t^5+3t^4-2t^3+3t^2-3t+1 \overset{?}{\ge} 0 \left(t=\frac{b}{c}\right) \Leftrightarrow (t-1)^4(t^2+t+1) \overset{?}{\ge} 0$$

$$\to \text{true} \therefore \frac{2b^2+2c^2}{b+c} \ge \sqrt[3]{4b^3+4c^3} \text{ and analogs} \Rightarrow$$

$$\frac{2a^2+2b^2}{a+b}+\frac{2b^2+2c^2}{b+c}+\frac{2c^2+2a^2}{c+a} \ge \sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3}$$

$$\overset{\text{via (1)}}{\Rightarrow} \frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a} \ge \sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3}$$

$$\forall a, b, c > 0, \text{"="} \text{ iff } a=b=c \text{ (QED)}$$