ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a + b = ab$, then prove that :

$$a^b + b^a > 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a + b = ab \Rightarrow a = b(a-1) > 0 \Rightarrow a > 1 \text{ and similarly, } b = a(b-1) > 0$$

$$\Rightarrow b > 1 \therefore a, b > 1 \therefore a^b + b^a \overset{\text{Bernoulli}}{>} \big(1 + b(a-1)\big) + \big(1 + a(b-1)\big)$$

$$= 2 + 2ab - (a+b) \overset{a+b=ab}{=} = 2 + 2ab - ab \therefore a^b + b^a > 2 + ab \rightarrow (1)$$

$$\text{Now, } ab = a + b \overset{A-G}{\geq} 2\sqrt{ab} \Rightarrow \sqrt{ab} \geq 2 \Rightarrow ab \geq 4 \rightarrow (2)$$

$$\therefore (1), (2) \Rightarrow a^b + b^a > 6 \text{ (QED)}$$