ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\left(a^3+b^3+c^3\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge\frac{3}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

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Solution by Tapas Das-India

$$2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=$$

$$=2\left(\left(\frac{x}{y}+\frac{y}{x}\right)^3-3\,\frac{x}{y}\cdot\frac{y}{x}\left(\frac{x}{y}+\frac{y}{x}\right)\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=$$

$$=2\left(\frac{x}{y}+\frac{y}{x}\right)^3-6\left(\frac{x}{y}+\frac{y}{x}\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=2\left(\frac{x}{y}+\frac{y}{x}\right)^3+9\left(\frac{x}{y}+\frac{y}{x}\right)+2\quad(1)$$

$$\textit{we have to show } \left(a^3+b^3+c^3\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge\frac{3}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

$$or\ \left(3+\frac{a^3}{b^3}+\frac{a^3}{c^3}+\frac{b^3}{a^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{c^3}{b^3}\right)\ge\frac{3}{2}\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)$$

$$or 2\left(3+\frac{a^3}{b^3}+\frac{a^3}{c^3}+\frac{b^3}{a^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{c^3}{b^3}\right)-3\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)\ge 0$$

$$or\left(6+2\frac{a^3}{b^3}+2\frac{a^3}{c^3}+2\frac{b^3}{a^3}+2\frac{b^3}{c^3}+2\frac{c^3}{a^3}+2\frac{c^3}{b^3}\right)-3\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)\ge 0$$

$$or\left(2\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)-3\left(\frac{a}{b}+\frac{b}{a}\right)+2\right)+\left(2\left(\frac{a^3}{c^3}+\frac{c^3}{a^3}\right)-3\left(\frac{a}{c}+\frac{c}{a}\right)+2\right)$$

$$+\left(2\left(\frac{b^3}{c^3}+\frac{c^3}{b^3}\right)-3\left(\frac{b}{c}+\frac{c}{b}\right)+2\right)\overset{(1)}{\ge}0$$

$$or\left(2\left(\frac{a}{b}+\frac{b}{a}\right)^3-9\left(\frac{a}{b}+\frac{b}{a}\right)+2\right)+\left(2\left(\frac{a}{c}+\frac{c}{a}\right)^3-9\left(\frac{a}{c}+\frac{c}{a}\right)+2\right)$$

$$+\left(2\left(\frac{b}{c}+\frac{c}{a}\right)^3-9\left(\frac{b}{c}+\frac{c}{b}\right)+2\right)\ge 0\ (2)$$

$$Let\ p=\left(\frac{a}{b}+\frac{b}{a}\right)\overset{AM-GM}{\ge}2,\ q=\left(\frac{a}{c}+\frac{c}{a}\right)\overset{AM-GM}{\ge}2,\ r=\left(\frac{b}{c}+\frac{c}{b}\right)\overset{AM-GM}{\ge}2$$

$$From\ (2) we\ get\ \left(2p^3-9p+2\right)+\left(2q^3-9q+2\right)+\left(2r^3-9r+2\right)\ge 0$$

or

$$(p-2)(2p^2+4p-1)+(q-2)(2q^2+4q-1)+(r-2)(2r^2+4r-1)\ge 0\ true\ (as\ p,q,r\ge 2)$$

$$Equality\ holds\ for\ a=b=c$$