ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c \geq 0$ and $a+b+c=3$, then prove that :

$$\frac{a}{b^4+16}+\frac{b}{c^4+16}+\frac{c}{a^4+16} \geq \frac{5}{32}$$

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$$\frac{a}{b^4+16}+\frac{b}{c^4+16}+\frac{c}{a^4+16} \geq \frac{5}{32} \Leftrightarrow \frac{1}{16}.\sum_{cyc}\frac{a(16+b^4-b^4)}{b^4+16} \geq \frac{5}{32}$$

$$\overset{a+b+c=3}{\Leftrightarrow} \frac{6}{32}-\frac{5}{32} \geq \frac{1}{16}.\sum_{cyc}\frac{ab^4}{b^4+16} \Leftrightarrow \sum_{cyc}\frac{ab^4}{b^4+16} \leq \frac{1}{2} \rightarrow (1)$$

WLOG we may assume $b \geq 2$ and then : $0 \leq c+a \leq 1$ ($\because a+b+c=3$)

$$\Rightarrow 0 \leq c,a \leq 1 \rightarrow (i)$$

$$\text{Now, } \sum_{cyc}\frac{ab^4}{b^4+16} \leq \frac{1}{2} \Leftrightarrow \frac{(3-b-c)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+\frac{ca^4}{a^4+16} \leq \frac{1}{2}$$

$$\Leftrightarrow \frac{(3-b)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+c\left(\frac{a^4}{a^4+16}-\frac{b^4}{b^4+16}\right) \leq \frac{1}{2}$$

$$\Leftrightarrow \frac{(3-b)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)}-\frac{15b^4c}{(a^4+16)(b^4+16)} \leq \frac{1}{2}$$

$$\Leftrightarrow \frac{(3-b)b^4}{b^4+16}+\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)}+bc\left(\frac{c^3}{c^4+16}-\frac{15b^3}{(a^4+16)(b^4+16)}\right) \boxed{\overset{(\blacksquare)}{\leq}} \frac{1}{2}$$

$$\text{Let } F(b)=\frac{(3-b)b^4}{b^4+16}\ \forall\, b \in [2,3] \text{ and then : } F'(b)=\frac{-b^3(b^5+80b-192)}{(b^4+16)^2}$$

$$=\frac{-b^3(b^4+2b^3+4b^2+8b+96)}{(b^4+16)^2}=0 \text{ iff } b=2 \text{ and } F''(b)=-\frac{5}{4}<0$$

$\therefore F(b)$ *attains a maxima at* $b=2$ *and*

$$\because F(b) \text{ never attains a minima in } [2,3] \therefore F(b) \leq F(2)=\frac{1}{2} \Rightarrow \boxed{\frac{(3-b)b^4}{b^4+16} \overset{(\bullet)}{\leq} \frac{1}{2}}$$

$$\text{Now, via (i), } a \leq 1 \leq \frac{b}{2} \Rightarrow 16a^4-b^4 \leq 0 \text{ and } \because c \geq 0$$

$$\therefore \boxed{\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)} \overset{(\bullet\bullet)}{\leq} 0}$$

$$\text{Again, let } f(t)=\frac{t^3}{t^4+16}\ \forall\, b \in [0,3] \text{ and then : } f'(t)=\frac{t^2(48-t^4)}{(t^4+16)^2}=0 \text{ iff}$$

$$t=2.\sqrt[4]{3} \therefore f'(t) \geq 0\ \forall\, t \in \left[0,2.\sqrt[4]{3}\right] \text{ and } f'(t) \leq 0\ \forall\, t \in \left[2.\sqrt[4]{3},3\right]$$

$$\Rightarrow f(t) \text{ is } \uparrow \text{ on } \left[0,2.\sqrt[4]{3}\right] \text{ and is } \downarrow \text{ on } \left[2.\sqrt[4]{3},3\right] \therefore \forall\, t \in [0,1], f(t) \leq f(1) \text{ and}$$

$$\because 0 \le c \le 1 \therefore \frac{c^3}{c^4+16} \overset{(*)}{\le} \frac{1^3}{1^4+16} = \frac{1}{17}$$

$Also, f(b)$ is $\uparrow$ on $[2, 2.\sqrt[4]{3}]$ and is $\downarrow$ on $[2.\sqrt[4]{3}, 3]$ and $\left.\frac{b^3}{b^4+16}\right|_{b=2} = \frac{1}{4}$ and

$\left.\frac{b^3}{b^4+16}\right|_{b=3} = \frac{27}{97} > \frac{1}{4}$ and hence, we conclude that $\forall\, b \in [2,3], \frac{b^3}{b^4+16} \overset{(**)}{\ge} \frac{1}{4}$

$$\therefore (*), (**) \text{ and } 0 \le a \le 1 \Rightarrow \frac{c^3}{c^4+16} - \frac{15b^3}{(a^4+16)(b^4+16)} \le \frac{1}{17} - \frac{15}{17}.\frac{1}{4} = -\frac{11}{68}$$

$$< 0 \text{ and } \because bc \ge 0 \therefore \boxed{bc\left(\frac{c^3}{c^4+16} - \frac{15b^3}{(a^4+16)(b^4+16)}\right) \overset{(\bullet\bullet\bullet)}{\le} 0}$$

$$\therefore (\bullet) + (\bullet\bullet) + (\bullet\bullet\bullet) \Rightarrow (\blacksquare) \Rightarrow (1) \text{ is true } \therefore \frac{a}{b^4+16} + \frac{b}{c^4+16} + \frac{c}{a^4+16} \ge \frac{5}{32}$$

$\forall\, a, b, c \ge 0 \,|\, a+b+c = 3$, "=" iff $(b = 2, c = 0, a = 1)$ or $(c = 2, a = 0, b = 1)$ or $(a = 2, b = 0, c = 1)$ (QED)