ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\ge\frac{\sqrt{2}}{4}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\sum_{cyc}\frac{b^2}{b+c}=\sum_{cyc}\frac{b^2-c^2+c^2}{b+c}$$

$$=\sum_{cyc}(b-c)+\sum_{cyc}\frac{c^2}{b+c}=\sum_{cyc}\frac{c^2}{b+c}\Rightarrow\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}$$

$$=\frac{1}{2}\sum_{cyc}\frac{b^2}{b+c}+\frac{1}{2}\sum_{cyc}\frac{c^2}{b+c}=\sum_{cyc}\frac{b^2+c^2}{2(b+c)}=\sum_{cyc}\frac{\sqrt{b^2+c^2}.\sqrt{b^2+c^2}}{2(b+c)}$$

$$\ge\sum_{cyc}\frac{\sqrt{b^2+c^2}.\sqrt{\frac{(b+c)^2}{2}}}{2(b+c)}=\frac{\sqrt{2}}{4}.\sum_{cyc}\sqrt{b^2+c^2}\therefore\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\ge$$

$$\frac{\sqrt{2}}{4}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$