ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$ and $xy \geq 1$, then prove that :

$$\frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{xy+1} \geq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Clearing denominators and simplifying}, \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{xy+1} \geq \frac{3}{2}$$

$$\Leftrightarrow 2xy(x^2+y^2) + 2(x^2+y^2) + x + y \geq 3x^2y^2 + xy(x+y) + 4xy + 1 \rightarrow (1)$$

$$\boxed{\text{Case 1}}\ xy + 1 > x + y \text{ and we have : LHS of } (1) - \text{RHS of } (1)$$

$$= (2xy(x^2+y^2) - 4x^2y^2) + x^2y^2 + (2(x^2+y^2) - 4xy) + x + y - xy(x+y) - 1$$

$$= 2xy(x-y)^2 + 2(x-y)^2 + (x^2y^2 - 1) - xy(x+y) + x + y$$

$$\geq (xy+1)(xy-1) - (x+y)(xy-1) = (xy-1)\big(xy + 1 - (x+y)\big) \geq 0$$

$$\because xy \geq 1 \text{ and } xy + 1 > x + y \Rightarrow (1) \text{ is true}$$

$$\boxed{\text{Case 2}}\ x + y \geq xy + 1 \text{ and we have : LHS of } (1) - \text{RHS of } (1) =$$

$$2xy(x^2+y^2) + 2(x^2+y^2) - 3x^2y^2 - xy(x+y) - 3xy + (x + y - xy - 1)$$

$$\geq 2xy(x^2+y^2) + 2(x^2+y^2) - 3x^2y^2 - xy(x+y) - 3xy$$

$$\overset{A-G}{\geq} 2xy(x^2+y^2) + xy - 3x^2y^2 - xy(x+y) \overset{A-G}{\geq}$$

$$xy\left(2(x^2+y^2) + 1 - \frac{3}{4}(x+y)^2 - (x+y)\right)$$

$$\geq xy\left((x+y)^2 + 1 - \frac{3}{4}(x+y)^2 - (x+y)\right) = \frac{xy}{4}.\left((x+y)^2 - 4(x+y) + 4\right)$$

$$= \frac{xy}{4}.(x+y-2)^2 \geq 0 \Rightarrow (1) \text{ is true} \therefore \text{combining both cases},$$

$$(1) \text{ is true } \forall\, x, y > 0 \,\big|\, xy \geq 1 \therefore \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{xy+1} \geq \frac{3}{2}$$

$$\forall\, x, y > 0 \,\big|\, xy \geq 1, \text{"="} \text{ iff } x = y = 1 \text{ (QED)}$$