ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2$, then prove that :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 4(a+b+c)$$

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$$\frac{1}{1+a}\overset{a>0}{<}1\ \therefore \text{ we can set : }\frac{1}{1+a}=1-x\ (x>0 \text{ and } x<1)$$

$$\therefore a+1=\frac{1}{1-x}\Rightarrow a=\frac{1}{1-x}-1=\frac{x}{1-x}\to(1)$$

$$\text{Similarly, we set : }\frac{1}{1+b}=1-y \text{ and } \frac{1}{1+c}=1-z\ \therefore 2=\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$$

$$=1-x+1-y+1-z\Rightarrow x+y+z=1\to(\text{i})\ \therefore (1)\text{ and (i)}\Rightarrow a=\frac{x}{y+z}\text{ and}$$

$$\text{analogously, } b=\frac{y}{z+x}\text{ and } c=\frac{z}{x+y}\text{ and hence : }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 4(a+b+c)$$

$$\text{transforms into : }\sum_{cyc}\frac{y+z}{x}\ge 4\sum_{cyc}\frac{x}{y+z}=4\sum_{cyc}\frac{x+y+z-(y+z)}{y+z}$$

$$=4\left(\sum_{cyc}x\right)\sum_{cyc}\frac{1}{y+z}-12\Leftrightarrow\sum_{cyc}\left(\frac{y+z}{x}+1\right)+9\ge 4\left(\sum_{cyc}x\right)\sum_{cyc}\frac{1}{y+z}$$

$$\Leftrightarrow\frac{1}{xyz}.\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+9\ge 4\left(\sum_{cyc}x\right).\frac{\sum_{cyc}x^2+3\sum_{cyc}xy}{\prod_{cyc}(y+z)}$$

$$\Leftrightarrow\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+9xyz\right).\prod_{cyc}(y+z)\ge 4xyz\left(\sum_{cyc}x\right)\left(\left(\sum_{cyc}x\right)^2+\sum_{cyc}xy\right)$$

$$\overset{\text{expanding and simplifying}}{\Leftrightarrow}\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4+2\sum_{cyc}x^3y^3\ge 2xyz\sum_{cyc}x^3+6x^2y^2z^2$$

$$\to\text{true}\because\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4=\sum_{cyc}x^4y^2+\sum_{cyc}x^4z^2\overset{A-G}{\ge}\sum_{cyc}2x^4yz$$

$$=2xyz\sum_{cyc}x^3\text{ and }2\sum_{cyc}x^3y^3\overset{A-G}{\ge}6x^2y^2z^2\ \therefore\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 4(a+b+c)$$

$$\forall\, a,b,c>0\ \Big|\ \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2, ''='' \text{ iff } a=b=c=\frac{1}{2}\ \text{(QED)}$$