ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a^3 + b^3 + 6ab \le 8$, then prove that:

$$\frac{1}{a^2 + b^2} + \frac{1}{ab} \ge \frac{3}{2}$$

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$$8 \ge a^3 + b^3 + 6ab \overset{A-G}{\ge} 2ab.\sqrt{ab} + 6ab \Rightarrow x^3 + 4x^2 - 4 \le 0 \; \left(x = \sqrt{ab}\right)$$

$$\Rightarrow (x-1)(x+1)^2 \le 0 \Rightarrow x = \sqrt{ab} \le 1 \therefore ab \le 1 \to (1)$$

$$\text{Now, via Power} - \text{Mean Inequality}, a^3 + b^3 \ge 2\left(\frac{a^2 + b^2}{2}\right)^{\frac{3}{2}}$$

$$\therefore 8 \ge 2\left(\frac{a^2 + b^2}{2}\right)^{\frac{3}{2}} + 6ab \Rightarrow (4 - ab)^{\frac{2}{3}} \overset{(\bullet)}{\ge} \frac{a^2 + b^2}{2}$$

$$\text{Now, } \frac{1}{a^2 + b^2} + \frac{1}{ab} \ge \frac{3}{2} \Leftrightarrow a^2 + b^2 + ab \ge \frac{3}{2}ab(a^2 + b^2)$$

$$\Leftrightarrow ab \overset{(*)}{\ge} (3ab - 2)\left(\frac{a^2 + b^2}{2}\right) \text{ and if } ab \le \frac{2}{3}, \text{then : RHS of } (*) \le 0 < ab$$

$$= \text{LHS of } (*) \Rightarrow (*) \text{ is true (strict inequality)}$$

$$\text{We now focus on : } ab > \frac{2}{3} \text{ and } (\bullet) \Rightarrow \text{in order to prove } (*), \text{it suffices to prove :}$$

$$ab \ge (3ab - 2)(4 - ab)^{\frac{2}{3}} \Leftrightarrow t^3 \ge (4 - t)^2(3t - 2)^3 \; (t = ab)$$

$$\Leftrightarrow 243t^5 - 1134t^4 + 2051t^3 - 1800t^2 + 768t - 128 \le 0$$

$$\Leftrightarrow (t-1)^2(243t^3 - 648t^2 + 512t - 128) \le 0$$

$$\Leftrightarrow 125(243t^3 - 648t^2 + 512t - 128) \le 0$$

$$\Leftrightarrow 30375t^3 - 81000t^2 + 64000t - 16000 \le 0$$

$$\Leftrightarrow 1215(t-1)(5t-3)^2 - 567(5t-3)^2 - 395\left(t - \frac{2}{3}\right) - \frac{676}{3} \le 0$$

$$\to \text{true (strict inequality)} \because t = ab \overset{\text{via } (1)}{\le} 1 \text{ and } t > \frac{2}{3} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^2 + b^2} + \frac{1}{ab} \ge \frac{3}{2} \; \forall \, a, b > 0 \,\middle|\, a^3 + b^3 + 6ab \le 8, " = " \text{ iff } a = b = 1 \text{ (QED)}$$