ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca + abc = 4$, then prove that :

$$0 \le \sqrt{a} + \sqrt{b} + \sqrt{c} - \sqrt{abc} \le \frac{2}{\sqrt{abc}}$$

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$$\sum_{cyc}\left((2+b)(2+c)\right) - (2+a)(2+b)(2+c)$$

$$= 4 - (ab + bc + ca + abc) = 0 \therefore \sum_{cyc}\frac{1}{2+a} = 1 \to (m)$$

$$\text{Now, } \frac{1}{2+a} \overset{a>0}{<} \frac{1}{2} \therefore \text{we can set : } \frac{1}{2+a} = \frac{1}{2} - x \left(x > 0 \text{ and } x < \frac{1}{2}\right)$$

$$\therefore a + 2 = \frac{2}{1-2x} \Rightarrow a = \frac{2}{1-2x} - 2 = \frac{2x}{\frac{1}{2} - x} \to (1)$$

$$\text{Similarly, we set : } \frac{1}{2+b} = \frac{1}{2} - y \text{ and } \frac{1}{2+c} = \frac{1}{2} - z$$

$$\therefore 1 \overset{\text{via } (m)}{=} \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} = \frac{1}{2} - x + \frac{1}{2} - y + \frac{1}{2} - z \Rightarrow x + y + z = \frac{1}{2} \to (i)$$

$$\therefore (1) \text{ and } (i) \Rightarrow a = \frac{2x}{y+z} \text{ and analogously, } b = \frac{2y}{z+x} \text{ and } c = \frac{2z}{x+y}$$

$$\text{and hence : } \sqrt{a} + \sqrt{b} + \sqrt{c} - \sqrt{abc} \le \frac{2}{\sqrt{abc}} \text{ transforms into :}$$

$$\sum_{cyc}\sqrt{\frac{2x}{y+z}} \le \sqrt{\frac{8xyz}{\prod_{cyc}(y+z)}} + \frac{2.\sqrt{\prod_{cyc}(y+z)}}{\sqrt{8xyz}}$$

$$= \frac{8xyz + 2\prod_{cyc}(y+z)}{\sqrt{8xyz\prod_{cyc}(y+z)}} \Leftrightarrow \sum_{cyc}\sqrt{\frac{2x}{y+z}} \boxed{\overset{(*)}{\le}} \frac{4xyz + \prod_{cyc}(y+z)}{\sqrt{2xyz\prod_{cyc}(y+z)}}$$

$$\text{Now, } \sum_{cyc}\sqrt{\frac{2x}{y+z}} \overset{\text{CBS}}{\le} \sqrt{2\sum_{cyc}x}.\sqrt{\sum_{cyc}\frac{1}{y+z}} = \sqrt{2\sum_{cyc}x}.\sqrt{\frac{\left(\sum_{cyc}x\right)^2 + \sum_{cyc}xy}{\prod_{cyc}(y+z)}}$$

$$\overset{?}{\le} \frac{4xyz + \prod_{cyc}(y+z)}{\sqrt{2xyz\prod_{cyc}(y+z)}}$$

$$\Leftrightarrow 2.\sqrt{xyz\sum_{cyc}x}.\sqrt{\left(\sum_{cyc}x\right)^2 + \sum_{cyc}xy} \boxed{\underset{(**)}{\overset{?}{\le}}} 4xyz + \prod_{cyc}(y+z)$$

Assigning $y+z=A, z+x=B, x+y=C \Rightarrow A+B-C=2z>0, B+C-A=2x > 0$ and $C+A-B=2y>0 \Rightarrow A+B>C, B+C>A, C+A>B \Rightarrow A,B,C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} A = 2s \Rightarrow \sum_{cyc} x \overset{(2)}{=} s \Rightarrow x = s-A, y = s-B, z = s-C$

$\Rightarrow xyz \overset{(3)}{=} r^2 s$ and via such substitutions, $\sum_{cyc} xy = \sum_{cyc}(s-A)(s-B) = 4Rr + r^2$

$$\Rightarrow \sum_{cyc} xy \overset{(4)}{=} 4Rr + r^2 \therefore (2),(3),(4) \Rightarrow (**)$$

$$\Leftrightarrow 4r^2s^2(s^2+4Rr+r^2) \le (4r^2s+4Rrs)^2$$

$$\Leftrightarrow s^2+4Rr+r^2 \le 4(R+r)^2 = 4R^2+8Rr+4r^2 \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2$$

$\rightarrow$ true via Gerretsen $\Rightarrow (**) \Rightarrow (*)$ is true $\therefore \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} \le \frac{2}{\sqrt{abc}}$

Again, $0 \le \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} \Leftrightarrow \sqrt{abc}.\sum_{cyc}\frac{1}{\sqrt{bc}} \ge \sqrt{abc} \Leftrightarrow \sum_{cyc}\frac{1}{\sqrt{bc}} \overset{(***)}{\ge} 1$ and

$$\sum_{cyc}\frac{1}{\sqrt{bc}} = \sum_{cyc}\frac{1^{\frac{3}{2}}}{\sqrt{bc}} \overset{\text{Radon}}{\ge} \frac{3^{\frac{3}{2}}}{\sqrt{\sum_{cyc} ab}} \overset{ab+bc+ca+abc=4}{=} \frac{3^{\frac{3}{2}}}{\sqrt{4-abc}} \overset{-abc<0}{>} \frac{\sqrt{27}}{2} > 1$$

$\Rightarrow (***)$ is true (strict inequality) $\therefore \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} > 0$ and so,

$$0 < \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} \le \frac{2}{\sqrt{abc}} \ \forall\, a,b,c>0 \,\middle|\, ab+bc+ca+abc=4,$$

$'' = ''$ iff $a=b=c=1$ (QED)