ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca + abc = 4$, then prove that :

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \sqrt{2 - \sqrt{ab}} + \sqrt{2 - \sqrt{bc}} + \sqrt{2 - \sqrt{ca}}$$

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$$\sum_{\text{cyc}} \left((2+b)(2+c)\right) - (2+a)(2+b)(2+c)$$

$$= 4 - (ab + bc + ca + abc) = 0 \therefore \sum_{\text{cyc}} \frac{1}{2+a} = 1 \to (\text{m})$$

$$\text{Now, } \frac{1}{2+a} \overset{a>0}{<} \frac{1}{2} \therefore \text{we can set : } \frac{1}{2+a} = \frac{1}{2} - x \left(x > 0 \text{ and } x < \frac{1}{2}\right)$$

$$\therefore a + 2 = \frac{2}{1-2x} \Rightarrow a = \frac{2}{1-2x} - 2 = \frac{2x}{\frac{1}{2} - x} \to (1)$$

$$\text{Similarly, we set : } \frac{1}{2+b} = \frac{1}{2} - y \text{ and } \frac{1}{2+c} = \frac{1}{2} - z$$

$$\therefore 1 \overset{\text{via (m)}}{=} \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} = \frac{1}{2} - x + \frac{1}{2} - y + \frac{1}{2} - z \Rightarrow x + y + z = \frac{1}{2} \to (\text{i})$$

$$\therefore (1) \text{ and (i)} \Rightarrow a = \frac{2x}{y+z} \text{ and analogously, } b = \frac{2y}{z+x} \text{ and } c = \frac{2z}{x+y}$$

$$\text{and hence : } \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \sqrt{2 - \sqrt{ab}} + \sqrt{2 - \sqrt{bc}} + \sqrt{2 - \sqrt{ca}}$$

$$\text{transforms into : } \sum_{\text{cyc}} \sqrt{\frac{2x}{y+z} \cdot \frac{2y}{z+x}} \overset{(*)}{\le} \sum_{\text{cyc}} \sqrt{2 - \sqrt{\frac{2x}{y+z} \cdot \frac{2y}{z+x}}}$$

Assigning $y + z = A, z + x = B, x + y = C \Rightarrow A + B - C = 2z > 0, B + C - A = 2x > 0$ and $C + A - B = 2y > 0 \Rightarrow A + B > C, B + C > A, C + A > B \Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{\text{cyc}} x = \sum_{\text{cyc}} A = 2s \Rightarrow \sum_{\text{cyc}} x = s \Rightarrow x = s - A, y = s - B, z = s - C$$

Via such substitutions, $(*)$ becomes :

$$2\sum_{\text{cyc}} \sqrt{\frac{(s-A)(s-B)}{AB}} \le \sum_{\text{cyc}} \sqrt{2 - 2\sqrt{\frac{(s-A)(s-B)}{AB}}}$$

$$\Leftrightarrow 2\sum_{cyc}\sin\frac{\alpha}{2}\boxed{\overset{(**)}{\leq}}\sum_{cyc}\sqrt{2-2\sin\frac{\alpha}{2}}\ (\alpha,\beta,\gamma\to \textit{angles of triangle with sides } A,B,C)$$

$$\text{Let } f(x)=\sqrt{2-2\sin\frac{x}{2}}-2\sin\frac{x}{2}+\left(x-\frac{\pi}{3}\right).\frac{3\sqrt{3}}{4}\ \forall\, x\in(0,\pi)\ \textit{and then}:$$

$$f'(x)=\frac{\sqrt{27}}{4}-\left(\cos\frac{x}{2}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)$$

$$=\frac{\sqrt{27}}{4}-\left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)\to(2)$$

$$\text{Now, }\frac{\sqrt{27}}{4}\overset{?}{\leq}\left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)$$

$$\Leftrightarrow\frac{\sqrt{27}}{4}\overset{?}{\leq}\left(\sqrt{1-t^2}\right)\left(1+\frac{1}{2\sqrt{2-2t}}\right)\ \left(t=\sin\frac{x}{2}\right)$$

$$\Leftrightarrow\frac{27}{16(1-t^2)}\overset{?}{\leq}1+\frac{1}{4(2-2t)}+\frac{1}{\sqrt{2-2t}}$$

$$\Leftrightarrow\frac{27}{16(1-t^2)}-1-\frac{1}{4(2-2t)}\overset{?}{\leq}\frac{1}{\sqrt{2-2t}}\Leftrightarrow\frac{9-2t+16t^2}{16(1-t^2)}\overset{?}{\leq}\frac{1}{\sqrt{2-2t}}$$

$$\Leftrightarrow\frac{(9-2t+16t^2)^2}{256(1-t^2)^2}\overset{?}{\leq}\frac{1}{2-2t}\ (\because 9>2>2t\Rightarrow 9-2t+16t^2>0)$$

$$\Leftrightarrow 128(1-t)(1+t)^2\overset{?}{\geq}(9-2t+16t^2)^2$$

$$\Leftrightarrow 256t^4+64t^3+420t^2-164t-47\overset{?}{\leq}0$$

$$\Leftrightarrow(2t-1)(128t^3+96t^2+258t+47)\overset{?}{\leq}0\Leftrightarrow t\overset{?}{\leq}\frac{1}{2}$$

$$\therefore\boxed{\frac{\sqrt{27}}{4}\leq\text{ or }\geq\left(\sqrt{1-t^2}\right)\left(1+\frac{1}{2\sqrt{2-2t}}\right)\ \textit{according as } t\leq\frac{1}{2}\text{ or } t\geq\frac{1}{2}}\ \textit{and } t\geq\frac{1}{2}$$

$$\Rightarrow 1>\sin\frac{x}{2}\geq\frac{1}{2}\Rightarrow\frac{\pi}{2}>\frac{x}{2}\geq\frac{\pi}{6}\Rightarrow\frac{\pi}{3}\leq x<\pi\ \textit{and } t=\sin\frac{x}{2}\leq\frac{1}{2}\Rightarrow 0<x\leq\frac{\pi}{3}$$

$$\therefore\frac{\sqrt{27}}{4}\leq\left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)\ \forall\, x\in\left(0,\frac{\pi}{3}\right]\ \textit{and}$$

$$\frac{\sqrt{27}}{4}\geq\left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)\ \forall\, x\in\left[\frac{\pi}{3},\pi\right)$$

$$\overset{\text{via }(2)}{\Rightarrow} f'(x) \le 0 \ \forall\, x \in \left(0, \frac{\pi}{3}\right] \text{ and } f'(x) \ge 0 \ \forall\, x \in \left[\frac{\pi}{3}, \pi\right) \Rightarrow f(x) \text{ is } \downarrow \text{ on } \left(0, \frac{\pi}{3}\right]$$

$$\text{and } f(x) \text{ is } \uparrow \text{ on } \left[\frac{\pi}{3}, \pi\right) \Rightarrow f(x) \ge f\left(\frac{\pi}{3}\right) = 0 \ \forall\, x \in (0, \pi)$$

$$\therefore \boxed{\sqrt{2 - 2\sin\frac{x}{2}} - 2\sin\frac{x}{2} \ge \left(\frac{\pi}{3} - x\right).\frac{3\sqrt{3}}{4} \ \forall\, x \in (0, \pi)}$$

$$\Rightarrow \sum_{\text{cyc}} \sqrt{2 - 2\sin\frac{\alpha}{2}} - 2\sum_{\text{cyc}} \sin\frac{\alpha}{2} \ge \left(3.\frac{\pi}{3} - (\alpha + \beta + \gamma)\right).\frac{3\sqrt{3}}{4} = (\pi - \pi).\frac{3\sqrt{3}}{4} = 0$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \sqrt{2 - \sqrt{ab}} + \sqrt{2 - \sqrt{bc}} + \sqrt{2 - \sqrt{ca}}$$

$$\forall\, a, b, c > 0 \,\big|\, ab + bc + ca + abc = 4, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$