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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a\mathrm{bc} = 1$, then prove that :

$$\frac{a}{\mathrm{b}}+\frac{\mathrm{b}}{\mathrm{c}}+\frac{\mathrm{c}}{a} \geq \frac{3}{2}(a+\mathrm{b}+\mathrm{c}-1)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{\mathrm{b}}+\frac{\mathrm{b}}{\mathrm{c}}+\frac{\mathrm{c}}{a} \geq \frac{3}{2}(a+\mathrm{b}+\mathrm{c}-1) \overset{a=\frac{1}{\mathrm{bc}}}{\Leftrightarrow} \frac{1}{\mathrm{b}^2\mathrm{c}}+\frac{\mathrm{b}}{\mathrm{c}}+\mathrm{bc}^2 \geq \frac{3}{2}\left(\frac{1}{\mathrm{bc}}+\mathrm{b}+\mathrm{c}-1\right)$$

$$\Leftrightarrow \frac{1+\mathrm{b}^3+\mathrm{b}^3\mathrm{c}^3}{\mathrm{b}^2\mathrm{c}} \geq \frac{3}{2}\left(\frac{1+\mathrm{b}^2\mathrm{c}+\mathrm{bc}^2-\mathrm{bc}}{\mathrm{bc}}\right)$$

$$\Leftrightarrow \boxed{2\mathrm{b}^3\mathrm{c}^3+2+2\mathrm{b}^3+3\mathrm{b}^2\mathrm{c} \overset{(*)}{\geq} 3\mathrm{b}+3\mathrm{b}^3\mathrm{c}+3\mathrm{b}^2\mathrm{c}^2}$$

$\boxed{\text{Case 1}}$ $\mathrm{bc} \geq 1$ and $\boxed{\text{Case 1a}}$ $\mathrm{b} \geq 1$ and $\because \mathrm{b}^3\mathrm{c}^3+\mathrm{b}^3+\mathrm{b}^3 \overset{A-G}{\geq} 3\mathrm{b}^3\mathrm{c}$

$$\therefore \text{it remains to prove} : \mathrm{b}^3\mathrm{c}^3+2+3\mathrm{b}^2\mathrm{c} \geq 3\mathrm{b}+3\mathrm{b}^2\mathrm{c}^2$$

$$\Leftrightarrow 3\mathrm{b}(\mathrm{bc}-1)+(\mathrm{bc}-1)(\mathrm{b}^2\mathrm{c}^2-2\mathrm{bc}-2) \geq 0$$

$$\Leftrightarrow (\mathrm{bc}-1)(\mathrm{b}^2\mathrm{c}^2-2\mathrm{bc}-2+3\mathrm{b}) \geq 0; \text{but} \because \mathrm{bc}-1 \geq 0 \text{ and } 3\mathrm{b} \geq 3$$

$$\therefore (\mathrm{bc}-1)(\mathrm{b}^2\mathrm{c}^2-2\mathrm{bc}-2+3) = (\mathrm{bc}-1)^3 \geq 0 \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 1b}}$ $\mathrm{b} \leq 1$ and $\because 2\mathrm{b}^3\mathrm{c}^3-3\mathrm{b}^2\mathrm{c}^2+1 = (\mathrm{bc}-1)^2(2\mathrm{bc}+1) \geq 0 \therefore$ it remains to prove : $2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}^2\mathrm{c}(1-\mathrm{b}) \geq 0$; but $\because \mathrm{bc} \geq 1$ and $1-\mathrm{b} \geq 0$

$$\therefore 2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}^2\mathrm{c}(1-\mathrm{b}) \geq 2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}(1-\mathrm{b}) = 2\mathrm{b}^3-3\mathrm{b}^2+1$$

$$= (\mathrm{b}-1)^2(2\mathrm{b}+1) \geq 0 \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $\mathrm{bc} \leq 1$ and $\boxed{\text{Case 2a}}$ $\mathrm{b} \leq 1 \wedge \mathrm{c} \geq 1$ and $\because 2\mathrm{b}^3\mathrm{c}^3-3\mathrm{b}^2\mathrm{c}^2+1$ $= (\mathrm{bc}-1)^2(2\mathrm{bc}+1) \geq 0 \therefore$ it remains to prove : $2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}^2\mathrm{c}(1-\mathrm{b})$ ≥ 0; but $\because \mathrm{c} \geq 1$ and $1-\mathrm{b} \geq 0 \therefore 2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}^2\mathrm{c}(1-\mathrm{b})$

$$\geq 2\mathrm{b}^3-3\mathrm{b}+1+3\mathrm{b}^2(1-\mathrm{b}) = (1-\mathrm{b})^3 \overset{\mathrm{b}\leq 1}{\geq} 0 \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2b}}$ $\mathrm{b} \geq 1 \wedge \mathrm{c} \leq 1$ and $\because 2\mathrm{b}^3\mathrm{c}^3-3\mathrm{b}^2\mathrm{c}^2+1 = (\mathrm{bc}-1)^2(2\mathrm{bc}+1) \geq 0$ $\therefore$ it remains to prove : $2\mathrm{b}^3-3\mathrm{b}+1 \geq 3\mathrm{b}^2\mathrm{c}(\mathrm{b}-1)$; but $\because \mathrm{bc} \leq 1$ and $\mathrm{b}-1 \geq 0$

$$\therefore 3\mathrm{b}^2\mathrm{c}(\mathrm{b}-1) \leq 3\mathrm{b}(\mathrm{b}-1) \overset{?}{\leq} 2\mathrm{b}^3-3\mathrm{b}+1 \Leftrightarrow 2\mathrm{b}^3-3\mathrm{b}^2+1 \overset{?}{\geq} 0$$

$$\Leftrightarrow (\mathrm{b}-1)^2(2\mathrm{b}+1) \overset{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2c}}$ $\mathrm{b}, \mathrm{c} \leq 1$ and we have : $\frac{1}{\mathrm{b}}-1, \frac{1}{\mathrm{c}}-1 \geq 0$ and we let : $x = \frac{1}{\mathrm{b}}-1$ and $\mathrm{y} = \frac{1}{\mathrm{c}}-1$ $(x, \mathrm{y} \geq 0) \therefore \mathrm{b} = \frac{1}{x+1}$ and $\mathrm{c} = \frac{1}{\mathrm{y}+1} \therefore (*)$ transforms into :

$$\frac{2}{(x+1)^3(\mathrm{y}+1)^3}+2+\frac{2}{(x+1)^3}+\frac{3}{(x+1)^2(\mathrm{y}+1)}$$

$$-\frac{3}{x+1}-\frac{3}{(x+1)^3(\mathrm{y}+1)}-\frac{3}{(x+1)^2(\mathrm{y}+1)^2} \geq 0 \overset{\text{simplifying}}{\Leftrightarrow}$$

$$2x^3y^3 + 6x^3y^2 + 3x^2y^3 + 6x^3y + 9x^2y^2 + 2x^3 + 9x^2y + 3xy^2 + y^3 + 3x^2 + 3xy$$
$$+3y^2 \geq 0 \to \text{true} \because x, y \geq 0 \Rightarrow (*) \text{ is true} \therefore \text{combining } all \text{ cases}, (*)\ \forall\, b, c > 0$$
$$\therefore \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{3}{2}(a + b + c - 1)\ \forall\, a, b, c > 0 \,\Big|\, abc = 1,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$