ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq 1, 0 < b \leq 1, 0 < c \leq 1$, *then prove that*:

$$\left(1+\frac{1}{abc}\right)(a+b+c) \geq 3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

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$$0 < a \leq 1 \Rightarrow \frac{1}{a} \geq 1 \text{ and we assign}: \frac{1}{a}-1 = x\ (x \geq 0) \therefore \frac{1}{a} = x+1$$

$$\Rightarrow a = \frac{1}{x+1} \text{ and similarly, we assign}: \frac{1}{b}-1 = y\ (y \geq 0) \text{ and } \frac{1}{c}-1 = z\ (z \geq 0)$$

$$\Rightarrow b = \frac{1}{y+1} \text{ and } c = \frac{1}{z+1} \text{ and via such transformation, } \left(1+\frac{1}{abc}\right)(a+b+c) \geq$$

$$3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \Leftrightarrow \left(1+\prod_{cyc}(x+1)\right)\left(\sum_{cyc}\frac{1}{x+1}\right) \geq 3+\sum_{cyc}x+3$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{x+1}+\sum_{cyc}\left((y+1)(z+1)\right) \geq 6+\sum_{cyc}x$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{x+1}+2\sum_{cyc}x+3+\sum_{cyc}xy \geq 6+\sum_{cyc}x \Leftrightarrow \sum_{cyc}\frac{1}{x+1}+\sum_{cyc}x+\sum_{cyc}xy \overset{(*)}{\geq} 3$$

$$\text{Now, } \sum_{cyc}\frac{1}{x+1}+\sum_{cyc}x+\sum_{cyc}xy \geq \sum_{cyc}x+\sum_{cyc}\frac{1}{x+1}\left(\because x,y,z \geq 0 \Rightarrow \sum_{cyc}xy \geq 0\right)$$

$$\overset{\text{Bergstrom}}{\geq} \sum_{cyc}x+\frac{9}{\sum_{cyc}x+3} = \frac{\left(\sum_{cyc}x\right)^2+3\sum_{cyc}x+9}{\sum_{cyc}x+3} = \frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x+3}+3 \geq 3$$

$$\Rightarrow (*) \text{ is true} \therefore \left(1+\frac{1}{abc}\right)(a+b+c) \geq 3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ \forall a,b,c \in (0,1],$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$