ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $a^7b^7(a^6+b^6) \geq 2$, then prove that :

$$a^2+b^2 \geq 2$$

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$$(a^2+b^2)^{10} \overset{?}{\geq} 512a^7b^7(a^6+b^6)$$

$$\Leftrightarrow (a^2+b^2)^9 \boxed{\underset{(*)}{\overset{?}{\geq}}} 512a^7b^7(a^4+b^4-a^2b^2)$$

$$\text{Now}, \forall\, a, b \in \mathbb{R}, (a-b)^2 \geq 0 \Rightarrow ab \leq \frac{a^2+b^2}{2}$$

$$\Rightarrow 512a^6b^6(a^4+b^4-a^2b^2).ab \leq 512a^6b^6(a^4+b^4-a^2b^2)\left(\frac{a^2+b^2}{2}\right)$$

$$(\because 512a^6b^6(a^4+b^4-a^2b^2) \geq 0) \overset{?}{\leq} (a^2+b^2)^9$$

$$\Leftrightarrow (a^2+b^2)^8 \overset{?}{\geq} 256a^6b^6(a^4+b^4-a^2b^2)\ (\because a^2+b^2 \geq 0)$$

$$\Leftrightarrow (x+y)^8 \boxed{\underset{(**)}{\overset{?}{\geq}}} 256x^3y^3(x^2+y^2-xy)\ (x=a^2, y=b^2)$$

$$\text{Now}, (x+y)^4 - 8xy(x^2+y^2) = (x^2+y^2+2xy)^2 - 8xy(x^2+y^2)$$
$$= (x^2+y^2)^2 - 4xy(x^2+y^2) + 4x^2y^2 = (x^2+y^2-2xy)^2 = (x-y)^4 \geq 0$$
$$\therefore (x+y)^4 \geq 8xy(x^2+y^2) \Rightarrow (x+y)^8 \geq 64x^2y^2(x^2+y^2)^2$$

$$(\because x=a^2, y=b^2 \Rightarrow x, y \geq 0 \Rightarrow xy \geq 0) \overset{?}{\geq} 256x^3y^3(x^2+y^2-xy)$$

$$\Leftrightarrow (x^2+y^2)^2 \overset{?}{\geq} 4xy(x^2+y^2-xy)$$

$$\Leftrightarrow (x^2+y^2)^2 - 4xy(x^2+y^2) + 4x^2y^2 \overset{?}{\geq} 0 \Leftrightarrow (x^2+y^2-2xy)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-y)^4 \overset{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore (a^2+b^2)^{10} \geq 512a^7b^7(a^6+b^6)$$
$$\geq 1024 \Rightarrow a^2+b^2 \geq 2\ \forall\, a, b \in \mathbb{R} \,\middle|\, a^7b^7(a^6+b^6) \geq 2,$$
$$'' = ''\ \text{ iff } (a=b=1) \text{ or } (a=b=-1)\ (\text{QED})$$