ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in \mathbb{R}$ and $xyz = 8$, then prove that :

$$\frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4} \geq 1$$

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$$\text{Let } x=\frac{2a}{b}, y=\frac{2b}{c}, z=\frac{2c}{a} \text{ and } \because x \neq 0, y \neq 0, z \neq 0 \therefore a \neq 0, b \neq 0, c \neq 0$$

$$\text{and via such transformation, } \frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4} \geq 1$$

$$\Leftrightarrow \frac{\frac{4a^2}{b^2}}{\frac{4a^2}{b^2}+\frac{4a}{b}+4}+\frac{\frac{4b^2}{c^2}}{\frac{4b^2}{c^2}+\frac{4b}{c}+4}+\frac{\frac{4c^2}{a^2}}{\frac{4c^2}{a^2}+\frac{4c}{a}+4} \geq 1 \Leftrightarrow \sum_{cyc}\frac{a^2}{a^2+ab+b^2} \geq 1 \rightarrow (1)$$

$$\text{Now, } a^2+ab+b^2=\frac{3}{4}(a+b)^2+\frac{1}{4}(a-b)^2 \text{ and so, if } (a^2+ab+b^2)=0, \text{ then :}$$

$$a=b \text{ and } a=-b \Rightarrow a=0, \text{ but } a \neq 0 \therefore a^2+ab+b^2 \neq 0 \Rightarrow a^2+ab+b^2>0$$

$$\text{and analogously, } (b^2+bc+c^2),(c^2+ca+a^2)>0 \therefore \sum_{cyc}\frac{a^2}{a^2+ab+b^2} \geq 1$$

$$\Leftrightarrow \sum_{cyc}\left(a^2(b^2+bc+c^2)(c^2+ca+a^2)\right) \geq \prod_{cyc}(a^2+ab+b^2)$$

$$\Leftrightarrow a^4b^2+b^4c^2+c^4a^2 \geq abc\sum_{cyc}ab^2 \rightarrow \text{true} \because \forall a,b,c \in \mathbb{R}, a^4b^2+b^4c^2+c^4a^2$$

$$\geq a^2b.b^2c+b^2c.c^2a+c^2a.a^2b=abc\sum_{cyc}ab^2 \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4} \geq 1$$

$$\forall x,y,z \in \mathbb{R} \,\middle|\, xyz=8, ''='' \text{ iff } x=y=z=2 \text{ (QED)}$$