ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$, $ab(a^6+64)(b^6+64) \geq 4400194256896$ then:

$$a^2+b^2 \geq 128$$

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$$\text{Let us assume : } a^2+b^2 < 128 \text{ and then :}$$
$$128 > a^2+b^2 \geq 2ab \Rightarrow ab < 64 \rightarrow (1)$$

$$\text{We have :}$$
$$4400194256896 \leq ab(a^6+64)(b^6+64) = ab(a^6b^6+64(a^6+b^6)+64^2)$$

$$= ab\left(a^6b^6+64\left((a^2+b^2)^3-3a^2b^2(a^2+b^2)\right)+64^2\right) \overset{\substack{\text{assumption} \\ \text{and} \\ \because -3a^2b^2(a^2+b^2) \leq -6a^3b^3}}{<}$$

$$ab(a^6b^6+64(128^3-6a^3b^3)+64^2)\ \left(\because ab \geq \frac{4400194256896}{(a^6+64)(b^6+64)} > 0\right)$$

$$\therefore t^7+64.128^3t-384t^4+64^2t > 4400194256896\ (t = ab)$$

$$\Rightarrow t^7-384t^4+134221824t-4400194256896 > 0$$

$$\Rightarrow (t-64)\begin{pmatrix} t^6+64t^5+4096t^4+261760t^3+16752640t^2 \\ +1072168960t+68753035264 \end{pmatrix} > 0 \Rightarrow t > 64$$

$(\because t = ab > 0)$ which is a contradiction to (1) $\therefore$ our assumption is incorrect

and hence we conclude that : $a^2+b^2 \geq 128$, $''=''$ iff $a = b = 8$ (QED)