ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ and $ab=1$, then prove that :

$$\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}\geq 2$$

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$$\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}=$$

$$=\frac{(a^3+2b^3)(a+2b)}{a^2(a^2+2b^2)(a+2b)}+\frac{(b^3+2a^3)(b+2a)}{b^2(b^2+2a^2)(b+2a)}\overset{\text{Reverse CBS}}{\geq}$$

$$\geq\frac{(a^2+2b^2)^2}{a^2(a^2+2b^2)(a+2b)}+\frac{(b^2+2a^2)^2}{b^2(b^2+2a^2)(b+2a)}=$$

$$=\frac{a^2+2b^2}{a^2(a+2b)}+\frac{b^2+2a^2}{b^2(b+2a)}\overset{ab=1}{=}\frac{1}{a+2b}+\frac{1}{b+2a}+\frac{2b^4}{a+2b}+\frac{2a^4}{b+2a}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{4}{3(a+b)}+\frac{2(a^2+b^2)^2}{3(a+b)}\geq\frac{4}{3(a+b)}+\frac{2.\frac{1}{4}.(a+b)^4}{3(a+b)}=$$

$$=\frac{t^3}{6}+\frac{4}{3t}\ (t=a+b)=\frac{t^4+8}{6t}\overset{?}{\geq}2$$

$$\Leftrightarrow t^4-12t+8\overset{?}{\geq}0\Leftrightarrow(t-2)(t^3+2t^2+4(t-2)+4)\overset{?}{\geq}0\to\text{true}\because t=a+b$$

$$\overset{A-G}{\geq}2\sqrt{ab}\overset{ab=1}{=}2\therefore\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}\geq 2,''='' \text{ iff } a=b=1\ (\text{QED})$$