ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that:

$$\frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}\geq 0$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}=\sum_{cyc}\frac{a^3}{a+3b}-\frac{1}{27}\cdot\sum_{cyc}\frac{(27b^3+a^3)-a^3}{a+3b}=$$

$$=\frac{28}{27}\cdot\sum_{cyc}\frac{a^4}{a^2+3ab}-\frac{1}{27}\cdot\sum_{cyc}(a^2+9b^2-3ab)\overset{\text{Bergstrom}}{\geq}$$

$$\geq\frac{28}{27}\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2+3\sum_{cyc}ab}-\frac{1}{27}\cdot\left(10\sum_{cyc}a^2-3\sum_{cyc}ab\right)=$$

$$=\frac{28x^2-(x+3y)(10x-3y)}{27(x+3y)}\left(x=\sum_{cyc}a^2, y=\sum_{cyc}ab\right)=\frac{18x^2-27xy+9y^2}{27(x+3y)}$$

$$=\frac{9(2x^2-3xy+y^2)}{27(x+3y)}=\frac{9(x-y)(2(x-y)+y)}{27(x+3y)}\geq 0 \because x\geq y>0$$

$$as\sum_{cyc}a^2\geq\sum_{cyc}ab\therefore\frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}\geq 0, ''='' \text{ iff } a=b=c \text{ (QED)}$$