ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}\ge 3(a^2+b^2+c^2)-(ab+bc+ca)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)=$$

$$=\frac{6b^3+14c^3-(6b^3+6bc^2-4b^2c+9b^2c+9c^3-6bc^2)}{2(2b+3c)}$$

$$\Rightarrow \frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)=\frac{5c(c^2-b^2)}{2(2b+3c)} \text{ and analogs}$$

$$\therefore \frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}\ge 3(a^2+b^2+c^2)-(ab+bc+ca)$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)\right)\ge 0$$

$$\Leftrightarrow \frac{c(c^2-b^2)}{2b+3c}+\frac{a(a^2-c^2)}{2c+3a}+\frac{b(b^2-a^2)}{2a+3b}\ge 0$$

$$\overset{\text{clearing denominators and simplifying}}{\Leftrightarrow} 2\sum_{cyc}a^4b+3\sum_{cyc}ab^4\overset{(*)}{\ge}5abc\sum_{cyc}ab$$

$$\text{Now, } 2\sum_{cyc}a^4b+3\sum_{cyc}ab^4=2abc\sum_{cyc}\frac{a^3}{c}+3abc\sum_{cyc}\frac{b^3}{c}\overset{\text{Holder}}{\ge}$$

$$2abc.\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}a}+3abc.\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}a}=\frac{5abc}{3}.\left(\sum_{cyc}a\right)^2\ge\frac{5abc}{3}.3\sum_{cyc}ab$$

$$=5abc\sum_{cyc}ab\Rightarrow(*)\text{ is true }\therefore\frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}$$

$$\ge 3(a^2+b^2+c^2)-(ab+bc+ca)\ \forall\, a,b,c>0,'' ='' \text{ iff } a=b=c \text{ (QED)}$$